THE CUBIC DIRAC EQUATION: SMALL INITIAL DATA IN $H^{\frac{1}{2}}(\mathbb{R}^2)$

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Abstract. Global well-posedness and scattering for the cubic Dirac equation with small initial data in the critical space $H^{\frac{1}{2}}(\mathbb{R}^2)$ is established. The proof is based on a sharp endpoint Strichartz estimate for the Klein-Gordon equation in dimension $n = 2$, which is captured by constructing an adapted systems of coordinate frames.

1. Introduction and main results

In this paper we continue our investigation initiated in [1] regarding the full range of Strichartz estimates available for the Klein-Gordon equation, with the particular goal of providing L^2L^∞ type estimates. As an application we prove global well-posedness and scattering for the cubic Dirac equation with small data in the critical space.

For fixed $m > 0$, we consider the (scalar) homogeneous Klein-Gordon equation

$$\Box u + m^2 u = 0, \qquad u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \text{ or } \mathbb{C}. \tag{1.1}$$

The validity of Strichartz estimates for solutions u of this equation is a fundamental and well-studied problem. In the low frequency regime, the dispersive properties of the Klein-Gordon equation are similar to the Schrödinger equation, i.e. the decay rate of the fundamental solution is $t^{-\frac{n}{2}}$. In the high frequency regime they are similar to the wave equation, i.e. the decay rate is $t^{-\frac{n-1}{2}}$. In the high frequency regime there is also a penalized Schrödinger-type decay: the fundamental solution localized at frequency 2^k decay as $2^k t^{-\frac{n}{2}}$; the penalization is due to the small curvature of the characteristic surface. If one is not concerned with sharp estimates, in the high frequency regime one could trade regularity for having access to the better decay $t^{-\frac{n}{2}}$. Such an approach severely limits the range of applications, in particular to low regularity nonlinear problems.

2010 *Mathematics Subject Classification.* 35Q41 (Primary); 35Q40, 35L02, 35L05 (Secondary).

Key words and phrases. Klein-Gordon equation, cubic Dirac equation, Strichartz estimate, well-posedness, scattering.

The decay rate of the fundamental solution plays a crucial role in determining the range of available Strichartz estimates. It is well-known that the endpoint Strichartz $L^2_t L^\infty_x$ estimate fails for the wave equation in dimensions $n = 3$ and for the Schrödinger equation in dimension $n = 2$, see [17, 25]. For the Klein-Gordon equation (1.1) in three dimensions, the endpoint $L^2_t L^\infty_x$ estimate does not fail if one allows for a loss of regularity, see [14]. However, the sharp $L^2_t L^\infty_x$ estimate (dictated by scaling) fails to hold true. In [1] we provided a microlocal replacement of the missing sharp endpoint $L^2_t L^\infty_x$ Strichartz estimate in dimension $n = 3$ by using adapted frames.

In dimension $n = 2$ the same problem becomes significantly more difficult since both endpoint Strichartz estimates for the wave equation, $L^4_t L^\infty_x$, and for the Schrödinger equation, $L^2_t L^\infty_x$, fail to hold. In this paper we address this problem by providing $L^2 L^\infty$ estimates in adapted frames. For the Klein-Gordon equation in dimension $n = 2$, to our best knowledge, these estimates are novel in literature.

Throughout the rest of this paper, we fix the physical dimension $n = 2$. In applications to nonlinear problems, see [13, 1] for the cubic Dirac equation in three dimensions, the endpoint Strichartz estimate and the $L^\infty L^2$ energy estimate imply a bilinear $L^2_{t,x}$ estimate via the toy scheme

$$\|u \cdot v\|_{L^2_{t,x}} \leq \|u\|_{L^2 L^\infty} \|v\|_{L^\infty L^2}.$$

Since the $L^2 L^\infty$ estimate will be established in adapted frames, energy estimates in similar frames are needed to recover the above $L^2_{t,x}$ bilinear estimate. As in dimension $n = 3$ in [1], combining the energy and the Strichartz estimate to derive a uniform L^2 estimate is only possible in presence of a null structure, see Section 3.

The idea to use adapted frames in order to find a replacement for the missing $L^2 L^\infty$ endpoint Strichartz estimate is due to Tataru [26], and was motivated by the Wave maps problem. In the context of the Schrödinger equation, this was done for solving the Schrödinger Map problem in two dimensions in [2]. Naively, one may expect that by using the structures in [26] and [2], one can address the same problem for the Klein-Gordon, but this is not the case. The reason is two-fold: there are no straight lines (zero curvature submanifolds) foliating the characteristic surface so as to emulate the Wave Equation construction; trading regularity in order to rely only on the Schrödinger equation would provide non-optimal estimates.

Instead, our current work builds on ideas from [26] and [2] and brings new ideas to provide a more complex construction well-adapted the geometry of the characteristic surface for the Klein-Gordon equation.

As an application, we study the cubic Dirac equation in dimension $n = 2$ at the critical regularity: Fix $M > 0$. Using the summation convention, the cubic Dirac equation for the spinor field $\psi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}^2$ is

$$(-i\gamma^\mu \partial_\mu + M)\psi = \langle \gamma^0 \psi, \psi \rangle \psi, \tag{1.2}$$

where $\gamma^\mu \in \mathbb{C}^{2\times 2}$ are the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{C}^2$.

The matrices γ^μ satisfy $\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2g^{\alpha\beta} I_2$, where $(g^{\alpha\beta}) = \mathrm{diag}(1, -1)$. By adapting the set of matrices, the equation (1.2) can be written in any spatial dimension. We refer the reader to [8, 21] for the physical background for this equation.

The n-dimensional version of (1.2) becomes critical in $H^{\frac{n-1}{2}}(\mathbb{R}^n)$ in the sense that it is (approximately) invariant under rescaling of solutions. In three dimensions the equation was studied extensively, see [7, 14, 13, 23, 5, 16] and references therein. The global well-posedness for small data in the critical space was established by the authors in [1].

In dimension $n = 2$ and $M \neq 0$, we are aware of only two results: [19, 20] where Pecher establishes local well-posedness of the equation with initial data in $H^s(\mathbb{R}^2)$ for $s > \frac{1}{2}$ and [3] where Bournaveas and Candy establish local well-posedness of the equation with initial data in $H^{\frac{1}{2}}(\mathbb{R}^2)$. To our best knowledge, no global well-posedness result is known so far. The case $M = 0$ has been settled in [3] where Bournaveas and Candy also prove global well-posedness and scattering for small initial data in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$, see more commentaries below about this case.

Our main result in this paper is

Theorem 1.1. *Let $M \neq 0$. The initial value problem associated to the cubic Dirac equation* (1.2) *is globally well-posed for small initial data $\psi(0) \in H^{\frac{1}{2}}(\mathbb{R}^2)$. Moreover, these solutions scatter to free solutions for $t \to \pm\infty$.*

For results in space dimension $n = 1$, see [15, 4].

A special case arises in the massless variant of the cubic Dirac equation, that is (1.2) with $M = 0$. A recent result of Bournaveas and Candy [3] provides the equivalent result of Theorem 1.1 for the case $M = 0$. Their strategy stems from the observation that the massless case carries similarities to the Wave Maps equation. The authors tailor their resolution spaces around the original ones introduced by Tataru

[26] in the context of Wave Maps. In order to overcome the Besov space obstacle, the authors of [3] used an idea from [1], i.e. adding a high modulation nonlinear structure of type $L_t^p L_x^2$ for certain $p < 2$. The authors of [3] also obtain a local in time result for $M \neq 0$ by treating the mass term $M\psi$ as a perturbation. However, from the perspective of obtaining a global in time result, the above strategy is limited to the case $M = 0$ since the resolution spaces for $M \neq 0$ were not known prior to the work in the present paper.

Our results here and the one in dimension $n = 3$ from [1] may seem orthogonal to the work of Bournaveas and Candy [3]. Indeed, we do not address directly the problem with $M = 0$. However by passing to the high frequency limit one can —at least formally— recoup the results for $M = 0$ since we work in the in the scale invariant space dictated by the wave part. We do not formalize this here and note that the approach in [3] is a more elegant and easier way to deal with this problem with $M = 0$. It is an instructive exercise is to check that, on fixed bounded time intervals, our structures become in the high frequency limit the ones used in [3] and originating in the work of Tataru [26].

We describe some of the key ideas involved in this paper. The Klein-Gordon waves travel with speed strictly less than 1, though in the high frequency limit the speed converges to 1. Our frames capture the speed variation of these waves as well as their directions, and this is why we work with two parameters: ω (angle) and λ (speed). Having a precise formulation on how the range of speed parameter λ depends on the frequency plays a crucial role in the argument.

The first system of frames we construct to recover an $L^2 L^\infty$ estimate stems from the one used [1]. An additional level of complexity is required due to the fact that once the high frequency waves enter the Schrödinger regime the decay rate fails to provide us with a classical $L_t^2 L_x^\infty$ estimate. To fix this issue we need a bi-parameter system of frames which depends both on ω (angle) and λ (speed).

The next problem arises from that the above system is well suited for most angular interactions, but fails near the parallel interactions (in fact it works at exact parallel interactions). Moreover, the null structure cannot fix this failure as usually is the case. To remedy this problem we construct another system of frames which is suited precisely to those angular scales and highlights a key geometrical property of wave interactions: waves with distinct frequencies travel with different speeds in the context of the Klein-Gordon equation.

The paper is organized is as follows: In the following subsection we introduce notation. Section 2 is devoted to endpoint Strichartz and energy estimates. In Section 3 we recall the null-structure of the

cubic Dirac equation. In Section 4 we construct function spaces for the nonlinear problem. In Section 5 we prove auxiliary bilinear and trilinear estimates. In Section 6 we prove the crucial nonlinear estimates and provide a proof of Theorem 1.1.

We point out that the notation, setup and general reductions in the present paper are adopted from [1]. Also, we will repeatedly refer to [1] for arguments which are similar in two and three dimensions. As indicated above, the analysis in this paper is significantly more involved, so it might be useful for the reader to take a look at [1], too.

1.1. **Notation.** Here, we repeat the notation from [1, Subsection 1.1] and adjust it to the case $n = 2$: We define $A \prec B$ by $A \leq B - c$ for some absolute constant $c > 0$. Also, we define $A \ll B$ by $A \leq dB$ for some absolute small constant $0 < d < 1$. Similarly, we define $A \lesssim B$ to be $A \leq eB$ for some absolute constant $e > 0$, and $A \approx B$ iff $A \lesssim B \lesssim A$.

Let $\mathrm{d}(M_1, M_2)$ denote the euclidean distance between the two sets $M_1, M_2 \subset \mathbb{R}^2$.

We set $\langle \xi \rangle_k := (2^{-2k} + |\xi|^2)^{\frac{1}{2}}$ for $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^2$, and write $\langle \xi \rangle := \langle \xi \rangle_0$.

Throughout the paper, let $\rho \in C_c^\infty(-2, 2)$ be a fixed smooth, even, cutoff satisfying $\rho(s) = 1$ for $|s| \leq 1$ and $0 \leq \rho \leq 1$. For $k \in \mathbb{Z}$ we define $\chi_k : \mathbb{R}^2 \to \mathbb{R}$, $\chi_k(y) := \rho(2^{-k}|y|) - \rho(2^{-k+1}|y|)$, such that $A_k := \operatorname{supp}(\chi_k) \subset \{y \in \mathbb{R}^2 \colon 2^{k-1} \leq |y| \leq 2^{k+1}\}$. Let $\tilde{\chi}_k = \chi_{k-1} + \chi_k + \chi_{k+1}$ and $\tilde{A}_k := \operatorname{supp}(\tilde{\chi}_k)$.

We denote by $P_k = \chi_k(D)$ and $\tilde{P}_k = \tilde{\chi}_k(D)$. Note that $P_k \tilde{P}_k = \tilde{P}_k P_k = P_k$. Further, we define $\chi_{\leq k} = \sum_{l=-\infty}^{k} \chi_l$, $\chi_{>k} = 1 - \chi_{\leq k}$ as well as the corresponding operators $P_{\leq k} = \chi_{\leq k}(D)$ and $P_{>k} = \chi_{>k}(D)$.

We denote by $\mathcal{K}_l$ a collection of spherical arcs (caps) of diameter 2^{-l} which provide a symmetric and finitely overlapping cover of the unit circle $\mathbb{S}^1$. Let $\omega(\kappa)$ to be the "center" of κ and let $\Gamma_\kappa \subset \mathbb{R}^2$ be the cone generated by κ and the origin, in particular $\Gamma_\kappa \cap \mathbb{S}^1 = \kappa$.

Further, let η_κ be smooth partition of unity subordinate to the covering of $\mathbb{R}^2 \setminus \{0\}$ with the cones Γ_κ, such that each η_κ is supported in $\frac{3}{2}\Gamma_\kappa$ and is homogeneous of degree zero and satisfies

$$|\partial_\xi^\beta \eta_\kappa(\xi)| \leq C_\beta 2^{l|\beta|} |\xi|^{-\beta}, \quad |(\omega(\kappa) \cdot \nabla)^N \eta_\kappa(\xi)| \leq C_N |\xi|^{-N}.$$

Let $\tilde{\eta}_\kappa$ with similar properties but slightly bigger support $2\Gamma_\kappa$, such that $\tilde{\eta}_\kappa \eta_\kappa = 1$. We define $P_\kappa = \eta_\kappa(D)$, $\tilde{P}_\kappa = \tilde{\eta}_\kappa(D)$. With $P_{k,\kappa} := \eta_\kappa(D)\chi_k(D)$ and $\tilde{P}_{k,\kappa} := \tilde{\eta}_\kappa(D)\tilde{\chi}_k(D)$, we obtain the angular decomposition

$$P_k = \sum_{\kappa \in \mathcal{K}_l} P_{k,\kappa}$$

and $P_{k,\kappa}\tilde{P}_{k,\kappa} = \tilde{P}_{k,\kappa}P_{k,\kappa} = P_{k,\kappa}$. We further define $A_{k,\kappa} = \mathrm{supp}(\eta_\kappa\chi_k)$ and $\tilde{A}_{k,\kappa} = \mathrm{supp}(\tilde{\eta}_\kappa\tilde{\chi}_k)$.

We define $\widehat{Q^\pm_m u}(\tau,\xi) = \chi_m(\tau\mp\langle\xi\rangle)\widehat{u}(\tau,\xi)$, and $\widehat{Q^\pm_{\leq m}u}(\tau,\xi) = \chi_{\leq m}(\tau\mp\langle\xi\rangle)\widehat{u}(\tau,\xi)$. We also define $\tilde{Q}^\pm_m = Q^\pm_{m-1} + Q^\pm_m + Q^\pm_{m+1}$. We set $B^\pm_{k,m}$ to be the Fourier support of $Q^\pm_m$, and $\tilde{B}^\pm_{k,m}$ to be the Fourier support of $\tilde{Q}^\pm_m$. We define $Q^\pm_{\prec m} = \sum_{l=-\infty}^{m-c} Q^\pm_l$ for a fixed large integer $c > 30$, and $Q^\pm_{\succeq m} = I - Q^\pm_{\prec m}$. Given $k \in \mathbb{Z}$, and $\kappa \in \mathcal{K}_l$ for some $l \in \mathbb{N}$ we set $B^\pm_{k,\kappa}$ to be the Fourier-support of $Q^\pm_{\prec k-2l}P_{k,\kappa}$. Similarly we define $\tilde{B}^\pm_{k,\kappa}$.

Given a pair (λ,ω) with $\lambda \in \mathbb{R}$ and $\omega = (\omega_1,\omega_2) \in \mathbb{S}^1$, we define $\omega^\perp = (-\omega_2,\omega_1)$ and the directions

$$\begin{aligned}\Theta = \Theta_{\lambda,\omega} =& \frac{1}{\sqrt{1+\lambda^2}}(\lambda,\omega),\\ \Theta^\perp = \Theta^\perp_{\lambda,\omega} =& \frac{1}{\sqrt{1+\lambda^2}}(-1,\lambda\omega),\\ \Theta_{0,\omega^\perp} =& (0,\omega^\perp).\end{aligned}$$

With respect to this basis, understanding the vectors $\Theta_{\lambda,\omega}$, $\Theta^\perp_{\lambda,\omega}$, $\Theta_{0,\omega^\perp}$ as column vectors, we introduce the new coordinates t_Θ, x_Θ, with $x_\Theta = (x^1_\Theta, x^2_\Theta)$, defined by

$$\begin{pmatrix} t_\Theta \\ x^1_\Theta \\ x^2_\Theta \end{pmatrix} = \begin{pmatrix} \Theta_{\lambda,\omega} & \Theta^\perp_{\lambda,\omega} & \Theta_{0,\omega^\perp} \end{pmatrix}^t \begin{pmatrix} t \\ x_1 \\ x_2 \end{pmatrix} \tag{1.3}$$

If $\lambda = 1$ we obtain the characteristic directions (null co-ordinates) as in [26, p. 42] and [24, p. 476]. However, our analysis requires more flexibility in the choice of the frames. For fixed $k \in \mathbb{Z}$ we define $\lambda(k) = (1+2^{-2k})^{-\frac{1}{2}}$.

2. Linear estimates

As in [1, Section 2], we recall that the decay rates of solutions to the linear wave equation and Klein-Gordon equation are determined by the principal curvatures of the characteristic hypersurfaces. This is well-known and we refer the reader to the list of references provided in [1, page 47, line 22] and the detailed discussion in [18, Section 2.5].

In [1] we started investigating the endpoint Strichartz estimate for the Dirac and Klein-Gordon equations in dimension $n = 3$. In this paper we continue our investigation in that direction in dimension $n = 2$. This requires a far more delicate theory since we have to deal with a missing endpoint Strichartz estimate for the Schrödinger part as well.

For convenience, we set $m = 1$ in the Klein-Gordon equation (1.1), which extends by rescaling to (1.1) with any $m \neq 0$. In this case, the solution is given by

$$u(t) = \frac{1}{2}(e^{it\langle D\rangle} + e^{-it\langle D\rangle})u_0 + \frac{1}{2i}(e^{it\langle D\rangle} - e^{-it\langle D\rangle})\frac{u_1}{\langle D\rangle}. \tag{2.1}$$

where $\langle D\rangle$ is the Fourier multiplier with symbol $\langle \xi\rangle$. Obviously, we need to study the propagator $e^{\pm it\langle D\rangle}$. For the sake of the exposition, we work out all the estimates for $e^{it\langle D\rangle}$, the estimates for $e^{-it\langle D\rangle}$ are then obtained by simply reversing time in the estimates for $e^{it\langle D\rangle}$.

2.1. **Endpoint L^2L^∞ type Strichartz estimate.** Our main result in this subsection provides endpoint Strichartz estimates for functions localized in frequency. The construction of the frame systems needed to capture these estimates is time-dependent, but the constants involved in the estimates are time independent.

We fix $r \in \mathbb{N}$, construct spaces that depend on r and provide uniform estimates on intervals $[-T, T]$ with $2^{r-1} \leq T \leq 2^r$. For $k \leq 99$ and $\omega \in \mathbb{S}^1$ we define the set

$$\Lambda_{k,\omega} = \left\{ i2^{-r}; i \in \mathbb{Z}, |i| \leq \frac{2^r}{\sqrt{1+2^{-2k-4}}} \right\} \times \{\omega\}$$

and

$$\|\phi\|_{\sum_{\Lambda_{k,\omega}} L^2_{t_\Theta} L^\infty_{x_\Theta}} := \inf_{\phi = \sum_{\Theta\in\Lambda_{k,\omega}} \phi_\Theta} \sum_{\Theta\in\Lambda_{k,\omega}} \|\phi_\Theta\|_{L^2_{t_\Theta} L^\infty_{x_\Theta}}.$$

Note that if $k_1 \leq k_2 \leq 99$ then $\Lambda_{k_1,\omega} \subset \Lambda_{k_2,\omega}$. One could be more precise about $\Lambda_{k,\omega}$, but this is not needed for low frequencies. However it is needed for high frequencies and this motivates the next definition.

For $k \geq 100$, and $\omega \in \mathbb{S}^1$ we define

$$\Lambda_{k,\omega} = \left\{ \frac{1}{\sqrt{1+m^{-2}}}; m \in 2^{2k-r-20}\mathbb{Z} \cap [2^{k-3}, 2^{k+3}] \right\} \times \{\omega\}$$

if $k < r + 20$, while if $k \geq r + 20$,

$$\Lambda_{k,\omega} = \{\lambda(k)\} \times \{\omega\}$$

Recall that $\lambda(k) = (1 + 2^{-2k})^{-\frac{1}{2}}$. We also define

$$\Omega_{k,\omega} = \{\lambda(k)\} \times \left\{ R^i\omega; i \in \mathbb{Z}, |i| \leq 2^{-k-8+r} \right\},$$

where R denotes a rotation by 2^{-r}. Note that the above set reduces to $\Omega_{k,\omega} = \{\lambda(k)\} \times \{\omega\}$ if $k + 8 > r$. These multiscale constructions, corresponding to large families of frames, are needed in the case $2^k \lesssim T$; in the case $T \lesssim 2^k$, single frames suffice.

For $\kappa \in \mathcal{K}_{k+10}$, we set $\Lambda_{k,\kappa} := \Lambda_{k,\omega(\kappa)}$ and $\Omega_{k,\kappa} := \Omega_{k,\omega(\kappa)}$.

Using these sets, we define

$$\|\phi\|_{\sum_{\Lambda_{k,\kappa}} L^2_{t_\Theta} L^\infty_{x_\Theta}} := \inf_{\phi=\sum_{\Theta\in\Lambda_{k,\kappa}}\phi_\Theta} \sum_{\Theta\in\Lambda_{k,\kappa}} \|\phi_\Theta\|_{L^2_{t_\Theta} L^\infty_{x_\Theta}}$$

$$\|\phi\|_{\sum_{\Omega_{k,\kappa}} L^2_{x^2_\Theta} L^\infty_{(t,x^1)_\Theta}} := \inf_{\phi=\sum_{\Theta\in\Omega_{k,\kappa}}\phi_\Theta} \sum_{\Theta\in\Omega_{k,\kappa}} \|\phi_\Theta\|_{L^2_{x^2_\Theta} L^\infty_{(t,x^1)_\Theta}}$$

We are ready to state the main result containing an effective replacement structure for the missing endpoint Strichartz estimates.

Theorem 2.1. *Let $r > 0$ and $T \in (0, 2^r]$.*

i) *For all $k \leq 99$, $\omega \in \mathbb{S}^1$ and $f \in L^2(\mathbb{R}^2)$ with $\mathrm{supp}(\widehat{f}) \subset \tilde{A}_{\leq k}$,*

$$\|1_{[-T,T]}(t)e^{it\langle D\rangle} f\|_{\sum_{\Lambda_{k,\omega}} L^2_{t_\Theta} L^\infty_{x_\Theta}} \lesssim \|f\|_{L^2}, \tag{2.2}$$

where the implicit constant does not depend on r and T.

ii) *For all $k \geq 100$, $\kappa \in \mathcal{K}_{k+10}$, and $f \in L^2(\mathbb{R}^2)$ with $\mathrm{supp}(\widehat{f}) \subset \tilde{A}_{k,\kappa}$,*

$$\|1_{[-T,T]}(t)e^{it\langle D\rangle} f\|_{\sum_{\Lambda_{k,\kappa}} L^2_{t_\Theta} L^\infty_{x_\Theta}} \lesssim \|f\|_{L^2}, \tag{2.3}$$

$$\|1_{[-T,T]}(t)e^{it\langle D\rangle} f\|_{\sum_{\Omega_{k,\kappa}} L^2_{x^2_\Theta} L^\infty_{(t,x^1)_\Theta}} \lesssim 2^{\frac{k}{2}} \|f\|_{L^2}, \tag{2.4}$$

where the implicit constants do not depend on r and T.

iii) *For all $k \geq 100$, $1 \leq l \leq k$, $\kappa_1 \in \mathcal{K}_l$ and $f \in L^2(\mathbb{R}^2)$ with $\mathrm{supp}(\widehat{f}) \subset \tilde{A}_{k,\kappa_1}$,*

$$\sum_{\kappa\in\mathcal{K}_k} \|1_{[-T,T]}(t)e^{it\langle D\rangle} \tilde{P}_\kappa f\|_{\sum_{\Lambda_{k,\kappa}} L^2_{t_\Theta} L^\infty_{x_\Theta}} \lesssim 2^{\frac{k-l}{2}} \|f\|_{L^2}. \tag{2.5}$$

where the implicit constant does not depend on r and T.

The estimate (2.2) is similar in nature to the corresponding estimate in [2, Lemma 3.4]. We highlight the similarities and the differences. By changing the variables and using that $|\lambda| \lesssim 1$ one passes from the frames used in [2, Lemma 3.4] to the ones used in this paper. We do not need to discriminate between the low frequencies and in this sense the estimate as listed here is suboptimal; one could easily restate it with a factor of $2^{\frac{k}{2}}$ for functions that are localized at frequency $\approx 2^k, k \leq 99$. The range of admissible λ is more carefully tracked here and this is why our version of Λ differs from the one used in [2, Lemma 3.4].

The rest of this subsection is devoted to the proof of Theorem 2.1. In order to prove (2.2) we consider the kernel

$$K_{\leq k}(t,x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{it\langle\xi\rangle} \tilde{\chi}^2_{\leq k}(|\xi|)\, d\xi, \tag{2.6}$$

for $k \leq 99$. The key estimates about this kernel are:

$$|K_{\leq k}(t,x)| \lesssim \langle t\rangle^{-1}, \tag{2.7}$$

$$|K_{\leq k}(t,x)| \lesssim_N \langle x\rangle^{-N}, \quad |x| \geq \frac{1}{\sqrt{1+2^{-2k-4}}}|t|. \tag{2.8}$$

Indeed, (2.7) is the standard decay rate for the Schrödinger kernel in dimension 2, which applies here because we truncate at low frequencies. (2.8) is obtained by using stationary phase type arguments, taking into account that the critical points of the phase function $\phi(\xi) = x\cdot\xi + t\langle\xi\rangle$ are contained inside the cone $|x| \leq \frac{1}{\sqrt{1+2^{-2k-2}}}|t|$.

For any $\omega \in \mathbb{S}^1$, we obtain the bound

$$|1_{[-T,T]}K_{\leq k}(t,x)| \lesssim_N \sum_{\Theta\in\Lambda_{k,\omega}} K_\Theta(t,x), \quad K_\Theta(t,x) = 2^{-r}\langle t_\Theta\rangle^{-N}.$$

This is obvious from (2.8) in the region of fast decay, and for fixed (t,x) in the region of slow decay we count the number of Θ such that $|t_\Theta| \lesssim 1$: If $|t| \lesssim 1$, every $\Theta \in \Lambda_{k,\omega}$ satisfies this, so the sum is of the size 1 which is ok in view of (2.7). In the case $|t| \gg 1$, the number of such Θ is $\approx 2^r t^{-1}$, so the sum is of size $\langle t\rangle^{-1}$, which is again fine because of (2.7).

From the expression of K_Θ we derive

$$\sum_{\Theta\in\Lambda_{k,\omega}} \|K_\Theta\|_{L^1_{t_\Theta}L^\infty_{x_\Theta}} \lesssim 1. \tag{2.9}$$

This suffices to prove (2.2). Indeed, by the TT^* argument and the duality:

$$\Big(\bigcap_{\Theta\in\Lambda_{k,\omega}} L^2_{t_\Theta}L^1_{x_\Theta}\Big)^* = \sum_{\Theta\in\Lambda_{k,\omega}} L^2_{t_\Theta}L^\infty_{x_\Theta}$$

the problem is reduced to proving $\|1_{[-T,T]}K_{\leq k}\|_{\sum_{\Lambda_{\leq k,\omega}} L^1_{t_\Theta}L^\infty_{x_\Theta}} \lesssim 1$, which follows from (2.9). A more complete formalization of this type of argument can be found in [2].

We continue the more delicate part of the argument, that is the analysis in high frequency with the aim of proving (2.3), (2.4) and (2.5). For $k\in\mathbb{Z}, k\geq 100$ we define

$$K_k(t,x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi}e^{it\langle\xi\rangle}\tilde{\chi}_k^2(|\xi|)\,d\xi. \tag{2.10}$$

and record the decay estimate

$$|K_k(t,x)| \lesssim 2^{2k}(1+2^k|(t,x)|)^{-\frac{1}{2}}\min(1,(1+2^k|(t,x)|)^{-\frac{1}{2}}2^k)). \tag{2.11}$$

This estimate appears in many places in literature, see for instance [18]. We provided a self-contained proof in [1] for dimension 3 which can be replicated almost verbatim for dimension 2 to give (2.11).

We define localized versions of the above kernel. For fixed $l \geq 1$ and $\kappa \in \mathcal{K}_l$ we define:

$$K_{k,\kappa}(t,x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{it\langle\xi\rangle} \tilde{\chi}_k^2(|\xi|)\tilde{\eta}_\kappa(\xi)\, d\xi. \tag{2.12}$$

$K_{k,\kappa}$ is the part of K_k localized in the angular cap κ. Also, we define

$$K^j_{k,\kappa}(t,x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{it\langle\xi\rangle} \alpha_j(2^{-k}|\xi|)\tilde{\chi}_k\tilde{\eta}_\kappa(\xi)\, d\xi, \tag{2.13}$$

where (α_j) is a smooth partition of unity with $\operatorname{supp}\alpha_j \subset \{(j-1)2^{-20} \leq |\xi| \leq (j+1)2^{-20}\}$. Obviously, we have

$$K_{k,\kappa}(t,x) = \sum_{j=2^{18}-1}^{2^{22}+1} K^j_{k,\kappa}. \tag{2.14}$$

The important decay properties of $K_{k,\kappa}$ and $K^j_{k,\kappa}$ are recorded in the following Proposition.

Proposition 2.2. *For all $k \in \mathbb{Z}, k \geq 100$, and $\kappa \in \mathcal{K}_{k+10}$,*

$$|K_{k,\kappa}(t,x)| \lesssim 2^k(1+2^{-k}|(t,x)|)^{-1}. \tag{2.15}$$

In addition, for $N = 1,2$, we have the following:

$$|K_{k,\kappa}(t,x)| \lesssim 2^k(1+|x^2_{k,\kappa}|)^{-N}, \textit{ if } |x^2_{k,\kappa}| \geq 2^{-k-9}|(t,x)|, \tag{2.16}$$

where $x^2_{k,\kappa} = x^2_{\Theta_{\lambda(k),\omega(\kappa)}}$. For $2^{18}-1 \leq j \leq 2^{22}+1$,

$$|K^j_{k,\kappa}(t,x)| \lesssim 2^k(1+2^k|t_{\lambda^j_k,\kappa}|)^{-N}, \textit{ if } |t_{\lambda^j_k,\kappa}| \geq 2^{-2k-8}|t|, \tag{2.17}$$

where $\lambda^j_k = 1/\sqrt{1+2^{-2k+40}j^{-2}}$ and $t_{\lambda^j_k,\kappa} = t_{\Theta_{\lambda^j_k,\omega(\kappa)}}$.

We remark that (2.16)-(2.17) hold with any $N \in \mathbb{N}$, but as stated it suffices for our purposes. Ideally one would like to have the estimate (2.17) for $K_{k,\kappa}$ is a similar form to (2.16) and skip the cumbersome $K^j_{k,\kappa}$ kernels. While available, such a formulation is not able to provide a strong exponent as above, see the factor 2^{-2k-8} in (2.17), and this would impact a key property of the set $\Lambda_{k,\kappa}$.

We now show how (2.3) follows from the above result. Fix $j \in [2^{18}-1, 2^{22}+1] \cap \mathbb{Z}$ and define

$$\Lambda^j_{k,\kappa} = \Big\{ \frac{1}{\sqrt{1+m^{-2}}}; m \in 2^{2k-r-20}\mathbb{Z} \cap [(j-1)2^{k-20}, (j+1)2^{k-20}] \Big\} \times \{\omega(\kappa)\}$$

We first make a few observations. The cardinality of each $\Lambda^j_{k,\kappa}$ is $2^{r-k+20} \approx 2^{-k}T$. This complicated construction is needed only for a certain range of frequencies: $2^k \lesssim T \approx 2^r$. If $k \geq r+20$, then we simply use a single set

$$\Lambda_{k,\kappa} = \left\{ \frac{1}{\sqrt{1+2^{-2k}}} \right\} \times \{\omega(\kappa)\},$$

and the arguments below simplify considerably: the claim (2.18) follows from Proposition 2.2 and the rest of the argument is identical.

Thus we focus below on the case $k \leq r+20$. For each $\Theta \in \Lambda^j_{k,\kappa}$ we define

$$K_\Theta(t,x) = 2^{2k}T^{-1}(1+2^k|t_\Theta|)^{-2}$$

and claim that

$$|K^j_{k,\kappa}(t,x)| \lesssim \sum_{\Theta\in\Lambda^j_{k,\kappa}} K_\Theta(t,x). \tag{2.18}$$

Since

$$\sum_{\Theta\in\Lambda^j_{k,\kappa}} \|K_\Theta\|_{L^1_{t_\Theta}L^\infty_{x_\Theta}} \lesssim |\Lambda^j_{k,\kappa}| \sup_{\Theta\in\Lambda^j_{k,\kappa}} \|K_\Theta\|_{L^1_{t_\Theta}L^\infty_{x_\Theta}} \lesssim 2^{-k}T\cdot 2^{2k}T^{-1}2^{-k} \lesssim 1.$$

we conclude with

$$\|K^j_{k,\kappa}\|_{\sum_{\Lambda^j_{k,\kappa}} L^1_{t_\Theta}L^\infty_{x_\Theta}} \lesssim 1.$$

By noting that $\Lambda_{k,\kappa} = \cup_j \Lambda^j_{k,\kappa}$, using (2.14) and the fact that j runs in a finite set, we obtain

$$\|K_{k,\kappa}\|_{\sum_{\Lambda_{k,\kappa}} L^1_{t_\Theta}L^\infty_{x_\Theta}} \lesssim 1.$$

which implies (2.3) by a TT^* argument similar to the one we used in the proof of (2.2).

We continue with the argument for (2.18). We start with a few observations, which in fact were the basis for the construction of the set $\Lambda^j_{k,\kappa}$:

P1: If $|t_{\lambda^j_k,\kappa}| \leq 2^{-2k-2}|(t,x)|$ then there exists $\Theta \in \Lambda^j_{k,\kappa}$ such that $|t_\Theta| \leq 2^{-k+2}$.

P2: If $|t_{\lambda^j_k,\kappa}| \geq 2^{-2k-2}|(t,x)|$ then $|t_{\lambda^j_k,\kappa}| \gtrsim |t_\Theta|$, for all $\Theta \in \Lambda^j_{k,\kappa}$.

As a first case, let (t,x) be such that $|t_{\lambda^j_k,\kappa}| \leq 2^{-2k-2}|(t,x)|$. From **P1** it follows that for each such (t,x) we estimate the number of $\Theta \in \Lambda^j_{k,\kappa}$ such that $|t_\Theta| \leq 2^{-k+2}$. If $\Theta_0 = (\lambda_0,\omega)$ is such a value, then any other such $\Theta = (\lambda,\omega)$ should satisfy $|(\lambda-\lambda_0)t| \leq 2^{-k+3}$. There are two subcases to consider next:

If $|t| \leq 2^k$, then since all $\Theta = (\lambda, \omega) \in \Lambda^j_{k,\kappa}$ satisfy $|\lambda - \lambda_0| \leq 2^{-2k+6}$ it follows that $|(\lambda - \lambda_0)t| \leq 2^{-k+6}$, hence the number of such Θ is $|\Lambda_{k,\kappa}| = 2^{-k}T$. Thus the sum on the right of (2.18) is estimated by $|\Lambda_{k,\kappa}| \cdot 2^{2k}T^{-1} = 2^k$ and this is the bound we have for the kernel $K_{k,\kappa}$.

If $|t| \geq 2^k$, we use that the discretization in $\Lambda^j_{k,\kappa}$ is at scale $2^{-k}T^{-1}$, it follows that the number of such λ is given by $\approx \frac{2^{-k}t^{-1}}{2^{-k}T^{-1}} = t^{-1}T$. The sum on the right of (2.18) is then $\gtrsim 2^{2k}T^{-1}t^{-1}T = 2^{2k}t^{-1}$ which is precisely the bound we have for the kernel $K_{k,\kappa}$.

Next we consider the second case where $|t_{k,\kappa}| \geq 2^{-2k-2}|(t,x)|$. We use **P2** : $|t_{k,\kappa}| \gtrsim |t_\Theta|$, for all $\Theta \in \Lambda^j_{k,\kappa}$. Thus $(1+2^k|t_\Theta|)^{-2} \gtrsim (1+2^k|t_{k,\kappa}|)^{-2}$ and the right hand side of (2.18) is $\gtrsim |\Lambda^j_{k,\kappa}| \cdot 2^{2k}T^{-1} \cdot (1+2^k|t_{k,\kappa}|)^{-2} = 2^k(1+2^k|t_{k,\kappa}|)^{-2}$ and this is the bound we have on $K^j_{k,\kappa}$ from (2.17). This finishes the proof of (2.3).

A similar argument using (2.16) proves (2.4). Note that the construction of the set $\Omega_{k,\kappa}$ was designed precisely to fit the corresponding **P1** and **P2** in this context: the angles considered in $\Omega_{k,\kappa}$ cover a neighborhood of $\omega(\kappa)$ size 2^{-k-8} which is double the size of the slow decay neighborhood described by (2.16).

Next we show how (2.5) follows from (2.3). Since there are $\approx 2^{k-l}$ caps $\kappa \in \mathcal{K}_k$ such that $P_\kappa f \neq 0$, we obtain from (2.3)

$$\begin{aligned}
&\sum_{\kappa \in \mathcal{K}_k} \|1_{[-T,T]}(t) e^{it\langle D\rangle} \tilde{P}_\kappa f\|_{\sum_{\Lambda_{k,\kappa}} L^2_{t_\Theta} L^\infty_{x_\Theta}} \\
&\lesssim 2^{\frac{k-l}{2}} \left(\sum_{\kappa \in \mathcal{K}_k} \|e^{it\langle D\rangle} \tilde{P}_\kappa f\|^2_{\sum_{\Lambda_{k,\kappa}} L^2_{t_\Theta} L^\infty_{x_\Theta}} \right)^{\frac{1}{2}} \\
&\lesssim 2^{\frac{k-l}{2}} \left(\sum_{\kappa \in \mathcal{K}_k} \|\tilde{P}_\kappa f\|^2_{L^2_x} \right)^{\frac{1}{2}} \lesssim 2^{\frac{k-l}{2}} \|f\|_{L^2_x}.
\end{aligned}$$

We end this section with the proof of (2.5).

Proof of Proposition 2.2. The following proof is very similar to [1]. We begin with the proof of (2.15). If $|(t,x)| \lesssim 2^k$ the claim follows from the fact that the domain of integration has measure $\approx 2^{2k-l} \approx 2^k$, otherwise the estimate follows from (2.11) and Young's inequality.

Next, we turn to the proof of (2.17). For compactness of notation, we write $\lambda = \lambda^j_k$ and $\Theta = \Theta_{\lambda^j_k, \omega(\kappa)}$. By rescaling it suffices to consider

$$B^j_{k,\kappa}(s,y) := \int_{\mathbb{R}^2} e^{iy\cdot\xi + is\langle\xi\rangle_k} \zeta_j(\xi) d\xi,$$

for $\zeta_j(\xi) = \alpha_j(|\xi|)\tilde{\chi}_1^2(|\xi|)\tilde{\eta}_\kappa(\xi)$, and to prove

$$|B^j_{k,\kappa}(s,y)| \lesssim_N 2^{-k}(1+|s_\Theta|)^{-N} \text{ if } |s_\Theta| \geq 2^{-2k-8}|s| \tag{2.19}$$

for $N = 1, 2$. If $|s_\Theta| \lesssim 1$, the estimate follows from the fact that the support of ζ_j has measure $\approx 2^{-k}$. Now, we assume $|s_\Theta| \gg 1$ and write $\phi(s,y,\xi) = y \cdot \xi + s\langle\xi\rangle_k$ Define $\partial_\omega = \omega \cdot \nabla_\xi$, $d_{\phi,\omega} := \frac{1}{i\partial_\omega \phi}\partial_\omega$ and $d^*_{\phi,\omega} := -\partial_\omega\Big(\frac{\cdot}{i\partial_\omega\phi}\Big)$. Integration by parts implies

$$\int_{\mathbb{R}^2} e^{i\phi(s,y,\xi)}\zeta_j(\xi)d\xi = \int_{\mathbb{R}^2} e^{i\phi(s,y,\xi)}(d^*_{\phi,\omega})^N\zeta_j(\xi))d\xi. \tag{2.20}$$

We will prove

$$|(d^*_{\phi,\omega})^N(\zeta_j)(\xi)| \lesssim_N |s_\Theta|^{-N}, \qquad N = 1, 2, \tag{2.21}$$

so that (2.19) follows from (2.20) and (2.21). Indeed, we observe that

$$\partial_\omega\phi(s,y,\xi) = s_{\lambda,\omega} + s\Big(\frac{\xi\cdot\omega}{\langle\xi\rangle_k} - \lambda\Big),$$

and in the domain of integration we have

$$\begin{aligned}\Big|\frac{\xi\cdot\omega}{\langle\xi\rangle_k} - \lambda\Big| &\leq \Big|\frac{1}{\sqrt{1+2^{-2k}|\xi|^{-2}}} - \lambda\Big| + \Big|\cos(\angle(\hat{\xi},\omega)) - 1\Big| \\ &\leq 2^{-2k-10} + 2^{-2k-10} \leq 2^{-2k-9},\end{aligned}$$

where we use that $(j-1)2^{-20} \leq |\xi| \leq (j+1)2^{-20}$ and $|\angle(\xi,\omega))| \leq 2^{-k-10}$. This implies

$$|\partial_\omega\phi(s,y,\xi)| \geq |s_\Theta| - |s|2^{-2k-9} \geq 2^{-1}|s_\Theta|.$$

In particular it follows that

$$|\frac{\partial_\omega\zeta}{\partial_\omega\phi}| \lesssim |s_\Theta|^{-1}. \tag{2.22}$$

where we used that $|\partial_\omega\zeta| \lesssim 1$. In addition, we have

$$\partial^2_\omega\phi(\xi) = \partial_\omega\Big(s\frac{\omega\cdot\xi}{\langle\xi\rangle_k}\Big) = s\left(\frac{\omega\cdot\omega}{\langle\xi\rangle_k} - \frac{(\omega\cdot\xi)^2}{\langle\xi\rangle_k^3}\right) = \frac{s}{\langle\xi\rangle_k}\left(1 - (\frac{\omega\cdot\xi}{\langle\xi\rangle_k})^2\right)$$

from which, using the above arguments, we conclude that in the domain of integration we have $|\partial^2_\omega\phi| \lesssim 2^{-2k}|s|$. This allows us to estimate

$$|\partial_\omega\Big(\frac{1}{\partial_\omega\phi}\Big)| \lesssim \frac{2^{-2k}|s|}{|\partial_\omega\phi|^2} \lesssim \frac{2^{-2k}|s|}{|s_\Theta|^2} \lesssim |s_\Theta|^{-1}.$$

From this and (2.22) we obtain (2.19) for $N = 1$. Now let $N = 2$ and compute

$$(d^*_{\phi,\omega^\perp})^2\zeta = \partial_\omega\Big(\frac{1}{\partial_\omega\phi}\partial_\omega\frac{\zeta}{\partial_\omega\phi}\Big) = \frac{\partial^2_\omega\zeta}{(\partial_\omega\phi)^2} - 3\frac{\partial_\omega\zeta\partial^2_\omega\phi}{(\partial_\omega\phi)^3} - \frac{\zeta\partial^3_\omega\phi}{(\partial_\omega\phi)^3} + 3\frac{\zeta(\partial^2_\omega\phi)^2}{(\partial_\omega\phi)^4}$$

We compute

$$\partial^3_\omega\phi = \frac{3s}{\langle\xi\rangle_k^5}\Big((\omega\cdot\xi)^3 - (\omega\cdot\xi)\langle\xi\rangle_k^2\Big) = \mathcal{O}(2^{-2k})|s|.$$

Recalling that $|\partial_\omega\phi| \geq \frac{1}{2}|s_\Theta| \gg 2^{-2k}$, $|\partial^2_\omega\phi| \lesssim 2^{-2k}$ and $|\partial^N_\omega\zeta| \lesssim_N 1$ we conclude that

$$|(d^*_{\phi,\omega^\perp})^N| \lesssim |s_\Theta|^{-2} + 2^{-2k}|s_\Theta|^{-3} + 2^{-4k}|s_\Theta|^{-4} \lesssim |s_\Theta|^{-2}.$$

This finishes the proof of (2.21) and, in turn, the proof of (2.17).

It remains to prove (2.16). We reset the definition of Θ to $\Theta = \Theta_{\lambda(k),\omega(\kappa)}$. As above, by rescaling it suffices to prove

$$|B_{k,\kappa}(s,y)| \lesssim_N 2^{-k}(1 + 2^{-k}|y^2_\Theta|)^{-N} \text{ if } |y^2_\Theta| \geq 2^{-k-8}|(s,y)| \tag{2.23}$$

for $N = 1, 2$, where we recall that $y^2_\Theta = y\cdot\omega^\perp$. If $|y^2_\Theta| \lesssim 2^k$, the estimate follows from the fact that the size of the support of integration is $\lesssim 2^{-k}$.

We now consider the case $|y^2_\Theta| \gg 2^k$. By replacing ω with $\omega^\perp$ in the above argument (see (2.20)), we obtain

$$\int_{\mathbb{R}^2} e^{i\phi(s,y,\xi)}\zeta(\xi)d\xi = \int_{\mathbb{R}^2} e^{i\beta\phi(s,y,\xi)}(d^*_{\phi,\omega^\perp})^N\zeta(\xi))d\xi. \tag{2.24}$$

As above, we claim

$$|(d^*_{\phi,\omega^\perp})^N(\zeta)(\xi)| \lesssim_N \Big(2^{-k}|y^2_\Theta|\Big)^{-N}, \qquad N = 1, 2. \tag{2.25}$$

Since the support of ζ has measure $\approx 2^{-k}$, (2.23) follows from (2.24) and (2.25).

We conclude the proof with the argument for (2.25). If ξ in the support of ζ then

$$\frac{\xi}{|\xi|} = (1 - c_1)\omega + c_2\omega^\perp, \qquad |c_1| \leq 2^{-2k-18}, |c_2| \leq 2^{-k-10}$$

and

$$|\frac{|\xi|}{\langle\xi\rangle_k} - \lambda| \leq 2^{-2k+4},$$

We compute

$$\partial_{\omega^\perp}\phi = \omega^\perp\cdot(y + s\frac{\xi}{\langle\xi\rangle_k}) = y^2_\Theta + c_2s\lambda + c_2s(\frac{|\xi|}{\langle\xi\rangle_k} - \lambda).$$

We have $|y^2_\Theta| \geq 2^{-k-9}|s| \geq 2|c_2 s\lambda|$, as well as $|y^2_\Theta| \geq 2^{-k-9}|(y,s)| \gg |c_2 s(\frac{|\xi|}{\langle\xi\rangle_k} - \lambda)|$. From these we conclude

$$|\partial_{\omega^\perp}\phi| \gtrsim |y^2_\Theta| \gg 2^k \tag{2.26}$$

and, using $|\partial_{\omega^\perp}\zeta| \lesssim 2^k$,

$$|\frac{\partial_{\omega^\perp}\zeta}{\partial_{\omega^\perp}\phi}| \lesssim 2^k |y^2_\Theta|^{-1}. \tag{2.27}$$

In addition, we have

$$\partial^2_{\omega^\perp}\phi(\xi) = \partial_{\omega^\perp}\left(r\frac{\omega^\perp\cdot\xi}{\langle\xi\rangle_k}\right) = s\left(\frac{\omega^\perp\cdot\omega^\perp}{\langle\xi\rangle_k} - \frac{(\omega^\perp\cdot\xi)^2}{\langle\xi\rangle_k^3}\right) = s(1+\mathcal{O}(2^{-k}))$$

within the support of ζ and we conclude

$$|\partial_{\omega^\perp}\Big(\frac{1}{\partial_{\omega^\perp}\phi}\Big)| \lesssim \frac{|s|}{|\partial_{\omega^\perp}\phi|^2} \lesssim \frac{|s|}{|y^2_\Theta|^2} \lesssim 2^k|y^2_\Theta|^{-1}.$$

From this and (2.27) we obtain (2.25) for $N=1$. Now we consider the case $N=2$ and compute

$$(d^*_{\phi,\omega^\perp})^2\zeta = \frac{\partial^2_{\omega^\perp}\zeta}{(\partial_{\omega^\perp}\phi)^2} - 3\frac{\partial_{\omega^\perp}\zeta\partial^2_{\omega^\perp}\phi}{(\partial_{\omega^\perp}\phi)^3} - \frac{\zeta\partial^3_{\omega^\perp}\phi}{(\partial_{\omega^\perp}\phi)^3} + 3\frac{\zeta(\partial^2_{\omega^\perp}\phi)^2}{(\partial_{\omega^\perp}\phi)^4}.$$

Further,

$$\partial^3_{\omega^\perp}\phi = \frac{3s}{\langle\xi\rangle_k^5}\Big((\omega^\perp\cdot\xi)^3 - (\omega^\perp\cdot\xi)\langle\xi\rangle_k^2\Big) = s\mathcal{O}(2^{-k}).$$

From (2.26) and $|\partial^2_{\omega^\perp}\phi| \lesssim |s|$ and $|\partial^N_{\omega^\perp}\zeta| \lesssim_N 2^{kN}$ it follows that

$$|(d^*_{\phi,\omega^\perp})^2| \lesssim 2^{2k}|y^2_\Theta|^{-2} + 2^k|y^2_\Theta|^{-3} + 2^{-k}|y^2_\Theta|^{-3} + |y^2_\Theta|^{-4} \lesssim 2^{2k}|y^2_\Theta|^{-2},$$

which completes the proof of (2.25) for $N=2$. □

2.2. **Energy estimates in the (λ,ω) frames.** Next, we prove energy estimates similar to [1, Subsection 2.2], but there will be important differences which we will point out below. At the end of the notation section we have introduced frames adapted to a pair (λ,ω) with $\lambda\in\mathbb{R}$ and $\omega\in\mathbb{S}^1$ and the new coordinates t_Θ, x_Θ. We denote by (τ_Θ,ξ_Θ) the corresponding Fourier variables which are given by

$$\begin{pmatrix}\tau_\Theta\\ \xi^1_\Theta\\ \xi^2_\Theta\end{pmatrix} = \begin{pmatrix}\Theta_{\lambda,\omega} & \Theta^\perp_{\lambda,\omega} & \Theta_{0,\omega^\perp}\end{pmatrix}\begin{pmatrix}\tau\\ \xi_1\\ \xi_2\end{pmatrix}$$

We also introduce here a fourth vector $\Theta^- = \Theta_{\lambda,-\omega}$ for reasons which will become apparent in the proof of the Theorem below. In the following theorem we set $B_{k,\kappa} = B^+_{k,\kappa}$ and $\tilde{B}_{k,\kappa} = \tilde{B}^+_{k,\kappa}$.

Theorem 2.3. *a) Let $99 \leq m = \min(j,k)$, $0 \leq l \leq m+10$ and $\kappa \in \mathcal{K}_l$. Let $\Theta = \Theta_{\lambda,\omega} \in \Lambda_{j,\omega}$. Assume $\alpha = \mathrm{d}(\omega,\kappa)$ satisfies $2^{-3-l} \leq \alpha \leq 2^{3-l}$ for $l \leq m+9$ and $\alpha \leq 2^{3-l}$ for $l = m+10$; if $j = 99$ then we consider only the last case. Define $\tilde{\alpha} = \max(\alpha, 2^{-m})$.*

i) *If $f \in L^2(\mathbb{R}^2)$ has the property that $\hat{f}$ is supported in $A_{k,\kappa}$, the following holds true*

$$\tilde{\alpha}\|e^{it\langle D\rangle} f\|_{L^\infty_{t_\Theta} L^2_{x_\Theta}} \lesssim \|f\|_{L^2}, \tag{2.28}$$

provided that $l \leq m-10$ or $l = m+10 \wedge |j-k| \geq 10$, and

$$\alpha^{\frac{1}{2}}\|e^{it\langle D\rangle} f\|_{L^\infty_{x^2_\Theta} L^2_{(t,x^1)_\Theta}} \lesssim \|f\|_{L^2}, \quad l \leq m+9. \tag{2.29}$$

ii) *Consider the inhomogeneous equation*

$$(i\partial_t + \langle D\rangle)u = g, \quad u(0) = 0, \tag{2.30}$$

where $\hat{g}$ is assumed to be supported in the set $B_{k,\kappa}$. If $g \in L^1_{t_\Theta} L^2_{x_\Theta}$, then the solution u satisfies the estimate

$$\tilde{\alpha}\|u\|_{L^\infty_{t_\Theta} L^2_{x_\Theta}} \lesssim \tilde{\alpha}^{-1}\|g\|_{L^1_{t_\Theta} L^2_{x_\Theta}}, \tag{2.31}$$

provided that $l \leq m-10$ or $l = m+10 \wedge |j-k| \geq 10$.

If $g \in L^1_{x^2_\Theta} L^2_{(t,x^1)_\Theta}$, then the solution u satisfies the estimate

$$\alpha^{\frac{1}{2}}\|u\|_{L^\infty_{x^2_\Theta} L^2_{(t,x^1)_\Theta}} \lesssim \alpha^{-\frac{1}{2}}\|g\|_{L^1_{x^2_\Theta} L^2_{(t,x^1)_\Theta}}, \quad l \leq m+9. \tag{2.32}$$

iii) *Under the hypothesis of Part ii) when $g \in L^1_{t_\Theta} L^2_{x_\Theta}$ the solution u can be written as*

$$u(t) = e^{it\langle D\rangle}\tilde{v}_0 + \int_{-\infty}^{\infty} u_s(t)\chi_{t_\Theta > s}\, ds \tag{2.33}$$

where $u_s(t) = e^{it\langle D\rangle} v_s$ (homogeneous solution in the original coordinates) and

$$\|\tilde{v}_0\|_{L^2_x} + \int_{-\infty}^{\infty} \|v_s\|_{L^2_x} ds \lesssim \alpha^{-1}\|g\|_{L^1_{t_\Theta} L^2_{x_\Theta}}. \tag{2.34}$$

In addition $\hat{v}_s$ and $\hat{\tilde{v}}_0$ are supported in $\tilde{A}_{k,\kappa}$.

A similar statement holds true when $g \in L^1_{x^2_\Theta} L^2_{(t,x^1)_\Theta}$.

A few remarks are in place about the statement of the above theorem. First, the statement (2.29) and the corresponding ones in part ii) and iii) hold true for all α with $2^{-3-l} \leq \alpha \leq 2^{3-l}$, in the sense that we do not need to restrict to $l \leq m+9$. The reason we did so in the statement is for the sake of conciseness. Nevertheless the statement (2.28) for $l = m+10$ does not require angular separation, thus covering the ranges skipped by the way we state (2.29).

What is important to note is that (2.28) fails somewhere in the range $m-9 \leq l \leq m+9$ in the sense that the energy estimates in the given frames "blow-up" and become useless. This is precisely the region where we need to use the estimates (2.29).

A careful reading reveals that in the case $|j-k| \leq 9$, and $l = m+10$ we did not provide any estimates. As noted above, one can continue estimates of type (2.29) and (2.32) for $l \geq m+10$, but these will not be helpful for our purposes.

Proof. i) **Proof of** (2.28)**.** We start with an almost verbatim repetition from [1, Proof of Theorem 2.4]: The space-time Fourier of $w(t,x) = e^{it\langle D\rangle} f(x)$ is given by the distribution $\mathcal{F}w = \hat{f} d\sigma$ where $d\sigma(\tau,\xi) = \delta_{\tau=\sqrt{|\xi|^2+1}}$ is comparable with the standard measure on the surface $\tau = \sqrt{|\xi|^2+1}$. We change the variables $(\tau,\xi) \to (\tau_\Theta, \xi_\Theta)$ and rewrite $\hat{f} d\sigma = F \delta_{\tau_\Theta = h(\xi_\Theta)}$; thus

$$\|F\|_{L^2_{\xi_\Theta}} \lesssim (1 + \|\nabla h\|_{L^\infty})^{\frac{1}{2}} \|f\|_{L^2} \tag{2.35}$$

where the L^∞ norms is taken on the support of F.

We now work out the details. The equation of the characteristic surface $\tau = \sqrt{|\xi|^2+1}$ can be rewritten as $\tau^2 - |\xi|^2 - 1 = 0$. In the new frame this takes the form

$$\frac{1}{\lambda^2+1}(\lambda\tau_\Theta - \xi^1_\Theta)^2 - \frac{1}{\lambda^2+1}(\tau_\Theta + \lambda\xi^1_\Theta)^2 - |\xi^2_\Theta|^2 - 1 = 0.$$

We solve this equation for τ_Θ, hence we rewrite it as follows

$$\frac{\lambda^2-1}{\lambda^2+1}(\tau_\Theta)^2 - \frac{4\lambda}{\lambda^2+1}\tau_\Theta\xi^1_\Theta + \frac{1-\lambda^2}{\lambda^2+1}(\xi^1_\Theta)^2 - |\xi^2_\Theta|^2 - 1 = 0. \tag{2.36}$$

The solutions of this quadratic equation are given by

$$\tau_\Theta = h^\pm(\xi_\Theta) = \frac{2\lambda\xi^1_\Theta \pm \sqrt{(\lambda^2+1)^2(\xi^1_\Theta)^2 + (\lambda^4-1)(|\xi^2_\Theta|^2+1)}}{\lambda^2-1}. \tag{2.37}$$

We will identify which one of the two solutions is the correct one. The positivity of the discriminant $\Delta_\Theta = (\lambda^2+1)^2(\xi^1_\Theta)^2 + (\lambda^4-1)(|\xi^2_\Theta|^2+1)$ is implicit, as we know a priori that (2.36) has at least one solution. We will come back shortly to these issues. We continue with the following

computation:

$$\begin{aligned}\frac{\partial h^\pm}{\partial \xi^1_\Theta} &= \frac{1}{\lambda^2-1}(2\lambda + \frac{(\lambda^2+1)^2\xi^1_\Theta}{\pm\sqrt{(\lambda^2+1)^2(\xi^1_\Theta)^2+(\lambda^4-1)(|\xi^2_\Theta|^2+1)}})\\ &= \frac{1}{\lambda^2-1}(2\lambda + \frac{(\lambda^2+1)^2\xi^1_\Theta}{(\lambda^2-1)\tau_\Theta - 2\lambda\xi^1_\Theta})\\ &= \frac{2\lambda\tau_\Theta + (\lambda^2-1)\xi^1_\Theta}{(\lambda^2-1)\tau_\Theta - 2\lambda\xi^1_\Theta} = -\frac{\xi^1_{\Theta^-}}{\tau_{\Theta^-}}\end{aligned}$$

In a similar manner we obtain $\nabla_{\xi^2_\Theta} h^\pm = (\lambda^2+1)\frac{\xi^2_\Theta}{\tau_{\Theta^-}}$, from which, using (2.35), it follows

$$\|e^{it\langle D\rangle} f\|_{L^\infty_{t_\Theta} L^2_{x_\Theta}} \lesssim \left(1 + \sup_{\xi \in A_{k,\kappa}} \frac{2^k}{|\tau_{\Theta^-}|}\right)^{\frac12} \|f\|_{L^2}. \tag{2.38}$$

To finish the argument we need a lower bound for $|\tau_{\Theta^-}|$. We provide below lower bounds for Δ_Θ and τ_{Θ^-} for $(\tau,\xi) \in B_{k,\kappa}$, as these more general bounds are needed in Part ii).

We need to consider a few cases: $j \le k-10$, $|j-k| \le 9$ and $j \ge k+10$. Since the computations are entirely similar, we will deal with $j \le k-10$ in detail. Here we have to consider two more cases: $l \le j-10$ and $l = j+10$.

Case 1: $l \le j-10$. For $(\tau,\xi) \in B_{k,\kappa}$ it holds that $\tau - \sqrt{|\xi|^2+1} = \epsilon(\tau,\xi)$ with $|\epsilon(\tau,\xi)| \le 2^{k-2l-10}$, hence

$$\begin{aligned}\tau_{\Theta^-} =& \lambda\tau - \xi\cdot\omega = \lambda\sqrt{|\xi|^2+1} + \lambda\epsilon - \xi\cdot\omega\\ =& |\xi|\left((\lambda-1)\sqrt{1+|\xi|^{-2}} + \sqrt{1+|\xi|^{-2}} - 1 + 1 - \frac{\xi\cdot\omega}{|\xi|} + \frac{\lambda\epsilon}{|\xi|}\right)\end{aligned}$$

We have the following: $|(1-\lambda)\sqrt{1+|\xi|^{-2}}| \le 2(1-\lambda) \le 2^{-2j+6} \le 2^{-2l-12}$ (since $\lambda \in \Lambda_j$), $|\sqrt{1+|\xi|^{-2}}-1| \le 2^{-2j-12} \le 2^{-2l-20}$, $2^{-2l-6} \le 1 - \frac{\xi\cdot\omega}{|\xi|} \le 2^{-2l+6}$ and $|\frac{\lambda\epsilon}{|\xi|}| \le 2^{-2l-8}$. From these we conclude that $2^{k-2l-10} \le \tau_{\Theta^-} \le 2^{k-2l+10}$; thus we conclude that $\tau_{\Theta^-} \approx 2^k\alpha^2$ and $\tau_{\Theta^-} \ge 2^{k-20}\alpha^2$.

In particular, using (2.38) we obtain (2.28). Since the solutions in (2.37) can be recast in the form $\tau_{\Theta^-} = \pm\sqrt{\Delta_\Theta}$ and we just proved that $\tau_{\Theta^-} > 0$ in $B_{k,\kappa}$, it follows that the solutions h^+ in (2.37) correspond to the choice of the surface $\tau = \sqrt{|\xi|^2+1}$.

We now continue with the more general bounds for Δ_Θ in the set $B_{k,\kappa}$. Since $|\tau - \langle\xi\rangle| \le 2^{k-2l-10}$, it follows that $|\tau^2 - |\xi|^2 - 1| \le 2^{2k-2l-8}$ or equivalently, $\tau^2 - |\xi|^2 - 1 = \epsilon(\tau,\xi)$ with $|\epsilon(\tau,\xi)| \le 2^{2k-2l-8}$. We

rewrite the equation in characteristic coordinates as above, to obtain

$$\tau_{\Theta^-}^2 = \Delta_\Theta + (1-\lambda^4)\epsilon$$

We have already shown that $\tau_{\Theta^-} \geq 2^{k-2l-10}$ and since $|(1-\lambda^4)\epsilon| \leq 2^{2k-2l-8}|1-\lambda| \leq 2^{2k-2l-8}2^{-2j+5} \leq 2^{2k-4l-23}$, it follows that $\Delta_\Theta \geq 2^{2k-4l-22} \approx 2^{2k}\alpha^4$ in $B_{k,\kappa}$. A similar argument proves $\Delta_\Theta \approx 2^{2k}\alpha^4$ in $B_{k,\kappa}$.

Case 2: $l = j+10$. For $(\tau,\xi) \in B_{k,\kappa}$ it holds that $\tau - \sqrt{|\xi|^2+1} = \epsilon(\tau,\xi)$ with $|\epsilon(\tau,\xi)| \leq 2^{k-2j-20}$, hence

$$\begin{aligned}\tau_{\Theta^-} =&\lambda\tau - \xi\cdot\omega = \lambda\sqrt{|\xi|^2+1} + \lambda\epsilon - \xi\cdot\omega \\ =&|\xi|\left((\lambda-1)\sqrt{1+|\xi|^{-2}} + \sqrt{1+|\xi|^{-2}} - 1 + 1 - \frac{\xi\cdot\omega}{|\xi|} + \frac{\lambda\epsilon}{|\xi|}\right)\end{aligned}$$

We have the following: $(1-\lambda)\sqrt{1+|\xi|^{-2}} \geq 1-\lambda \geq 2^{-2j-8}$ (since $\lambda \in \Lambda_j$), $|\sqrt{1+|\xi|^{-2}}-1| \leq 2^{-2j-12}$, $|1-\frac{\xi\cdot\omega}{|\xi|}| \leq 2^{-2j-12}$ and $|\frac{\lambda\epsilon}{|\xi|}| \leq 2^{-2j-12}$. From these we conclude that $-\tau_{\Theta^-} \approx 2^{k-2j}$ and also that $-\tau_{\Theta^-} \geq 2^{k-2j-10}$.

In particular, using (2.38) we obtain (2.28). Since the solutions in (2.37) can be recast in the form $\tau_{\Theta^-} = \pm\sqrt{\Delta_\Theta}$ and we just proved that $\tau_{\Theta^-} < 0$ in $B_{k,\kappa}$, it follows that the solutions h^- in (2.37) correspond to the choice of the surface $\tau = \sqrt{|\xi|^2+1}$.

We now continue with the more general bounds for Δ_Θ in the set $B_{k,\kappa}$. Since $|\tau - \langle\xi\rangle| \leq 2^{k-2j-30}$ hence $|\tau^2 - |\xi|^2 - 1| \leq 2^{2k-2j-28}$ or equivalently, $\tau^2 - |\xi|^2 - 1 = \epsilon(\tau,\xi)$ with $|\epsilon(\tau,\xi)| \leq 2^{2k-2j-28}$. We rewrite the equation in characteristic coordinates as above, to obtain

$$\tau_{\Theta^-}^2 = \Delta_\Theta + (1-\lambda^4)\epsilon$$

We have already shown that $\tau_{\Theta^-} \geq 2^{k-2j-10}$ and since $|(1-\lambda^4)\epsilon| \leq 2^{2k-2j-26}|1-\lambda| \leq 2^{2k-2j-26}2^{-2j+5} = 2^{2k-4j-21}$, it follows that $\Delta_\Theta \geq 2^{2k-2j-21}$ in $B_{k,\kappa}$. A similar argument proves $\Delta_\Theta \approx 2^{2k}\tilde{\alpha}^4$ in $B_{k,\kappa}$.

Although we decided to leave out the details of this argument in the cases $|j-k| \leq 9$ and $j \geq k+10$, we would like to point out a simple fact. If $j=k$, $\xi = 2^k\omega$ and $\epsilon = 0$, we obtain $\tau_{\Theta^-} = 0$. This highlights the reason why we cannot cover the case $l = m+10$ when $|j-k| \leq 9$.

Proof of (2.29)**.** We start as in the proof of (2.28) but with the goal of writing $\hat{f}d\sigma = F\delta_{\xi_\Theta^2 = h(\tau_\Theta,\xi_\Theta^1)}$. This gives the bound

$$\|F\|_{L^2_{\tau_\Theta,\xi_\Theta^1}} \lesssim (1+\|\nabla h\|_{L^\infty})^{\frac{1}{2}}\|f\|_{L^2} \tag{2.39}$$

where the L^∞ norm of ∇h is taken on the support of F.

We use the equation of the characteristic surface in the form (2.36) and solve this equation for $\xi'_{\lambda,\omega}$:

$$\xi^2_\Theta = \tilde{h}^\pm(\tau_\Theta, \xi^1_\Theta) = \pm\sqrt{\tilde{\Delta}_\Theta}. \tag{2.40}$$

where $\tilde{\Delta}_\Theta = \frac{1}{\lambda^2+1}(\lambda\tau_\Theta - \xi^1_\Theta)^2 - \frac{1}{\lambda^2+1}(\tau_\Theta + \lambda\xi^1_\Theta)^2 - 1$. Now,

$$\frac{\partial \tilde{h}^\pm}{\partial \tau_\Theta} = \frac{1}{\lambda^2+1}\frac{(\lambda^2-1)\tau_\Theta - 2\lambda\xi^1_\Theta}{\xi^2_\Theta} = \frac{1}{\lambda^2+1}\frac{\tau_{\Theta^-}}{\xi^2_\Theta}$$

In a similar manner we obtain $\frac{\partial \tilde{h}^\pm}{\partial \xi^1_\Theta} = (\lambda^2+1)\frac{\xi^1_{\Theta^-}}{\xi^2_\Theta}$, from which, using (2.39), it follows

$$\|e^{it\langle D\rangle} f\|_{L^\infty_{x^2_\Theta} L^2_{(t,x^1)_\Theta}} \lesssim \left(1 + \sup_{\xi \in A_{k,\kappa}} \frac{2^k}{|\xi^2_\Theta|}\right)^{\frac{1}{2}} \|f\|_{L^2}. \tag{2.41}$$

To finish the argument we use $|\xi^2_\Theta| = |\xi \cdot \omega^\perp| \approx 2^k \cdot \alpha$. As before, a direct computation shows that in the set $B_{k,\kappa}$ we have $|\xi^2_\Theta| \approx 2^k \cdot \alpha$ and $\tilde{\Delta}_\Theta \approx (2^k \cdot \alpha)^2$.

ii) and iii) The proofs of these estimates are entirely similar to the corresponding ones in [1]. The basic idea is that once the linear phenomenology is unraveled by (2.28) and (2.29), obtaining the energy type estimates is done in a similar manner: change the coordinates and estimate all quantities taking into account the localization in $B_{k,\kappa}$. Note that in part i) we upgraded some of our estimates to $B_{k,\kappa}$. $\square$

3. Reduction and Null structure of the cubic Dirac

The cubic Dirac equation (1.2) has a linear part with matrix coefficients. Below, we rewrite (1.2) as a new system which has two half Klein-Gordon equations as linear parts, see (3.3) below, and we identify a null-structure in the nonlinearity, similarly to the ideas for the Dirac-Klein-Gordon system presented in [6, Section 2 and 3] and adapted to the Cubic Dirac equation in dimension $n = 2$ in [19]. However, in contrast to the above mentioned papers, we keep the mass term inside the linear operator. The setup here is the two-dimensional equivalent of [1, Section 3] and we repeat the most important aspects.

Multiplying the cubic Dirac equation from the left with γ^0, we obtain

$$-i(\partial_t + \alpha \cdot \nabla + i\beta)\psi = \langle \psi, \beta\psi\rangle \beta\psi. \tag{3.1}$$

where $\beta = \gamma^0$ and $\alpha^j = \gamma^0\gamma^j$ and $\alpha \cdot \nabla = \alpha^j \partial_j$. The new matrices satisfy

$$\alpha^j\alpha^k + \alpha^k\alpha^j = 2\delta^{jk} I_2, \qquad \alpha^j\beta + \beta\alpha^j = 0. \tag{3.2}$$

Following [6] we decompose the spinor field relative to a basis of the operator $\alpha\cdot\nabla+i\beta$ with symbol $\alpha\cdot\xi+\beta$. Since $(\alpha\cdot\xi+\beta)^2=(|\xi|^2+1)I$, the eigenvalues are $\pm\langle\xi\rangle$. We introduce the projections $\Pi_\pm(D)$ with symbol

$$\Pi_\pm(\xi)=\frac{1}{2}[I\mp\frac{1}{\langle\xi\rangle}(\xi\cdot\alpha+\beta)].$$

As in [1], we slightly deviate from [6, formula (5)] by switching the sign in $\Pi_\pm$ for internal consistency purposes. The key identity is

$$-i(\alpha\cdot\nabla+i\beta)=\langle D\rangle(\Pi_-(D)-\Pi_+(D))$$

where $\langle D\rangle$ has symbol $\sqrt{|\xi|^2+1}$. The following identity, which can be verified easily at the level of the symbols, will be important in our computations:

$$\Pi_\pm(D)\beta=\beta(\Pi_\mp(D)\mp\frac{\beta}{\langle D\rangle}).$$

We then define $\psi_\pm=\Pi_\pm(D)\psi$ and split $\psi=\psi_++\psi_-$. By applying the operators $\Pi_\pm(D)$ to the cubic Dirac equation we obtain the following system of equations

$$\begin{cases}(i\partial_t+\langle D\rangle)\psi_+=-\Pi_+(D)(\langle\psi,\beta\psi\rangle\beta\psi)\\ (i\partial_t-\langle D\rangle)\psi_-=-\Pi_-(D)(\langle\psi,\beta\psi\rangle\beta\psi).\end{cases}\tag{3.3}$$

This system will replace (1.2) as the object of our research for the rest of the paper. It is obvious from the form of the operators $\Pi_\pm$ that $\|\psi\|_X\approx\|\psi_+\|_X+\|\psi_-\|_X$ for many reasonable function spaces X. In particular we use it for $X=H^{\frac12}(\mathbb{R}^2)$ so that we conclude that the initial data for (3.3) satisfies $\psi_\pm(0)\in H^{\frac12}(\mathbb{R}^2)$.

To reveal the null structure, we start with $\langle\psi,\beta\psi\rangle$ which, in our decomposition, is rewritten as

$$\begin{aligned}\langle\psi,\beta\psi\rangle=&\langle\Pi_+(D)\psi_+,\beta\Pi_+(D)\psi_+\rangle+\langle\Pi_-(D)\psi_-,\beta\Pi_-(D))\psi_-\rangle\\&+\langle\Pi_+(D)\psi_+,\beta\Pi_-(D)\psi_-\rangle+\langle\Pi_-(D)\psi_-,\beta\Pi_+(D)\psi_+\rangle\end{aligned}$$

Next we analyze the symbols of the bilinear operators above.

Lemma 3.1. *The following holds true*

$$\begin{aligned}\Pi_\pm(\xi)\Pi_\mp(\eta)&=\mathcal{O}(\angle(\xi,\eta))+\mathcal{O}(\langle\xi\rangle^{-1}+\langle\eta\rangle^{-1})\\ \Pi_\pm(\xi)\Pi_\pm(\eta)&=\mathcal{O}(\angle(-\xi,\eta))+\mathcal{O}(\langle\xi\rangle^{-1}+\langle\eta\rangle^{-1})\end{aligned}\tag{3.4}$$

Proofs of this result can be found [6] or [19] modulo the fact that the operators $\Pi_\pm$ there do not include the β factor; but this is accounted by the additional factor of $\mathcal{O}(\langle\xi\rangle^{-1}+\langle\eta\rangle^{-1})$ in the estimate above, see also [1, Lemma 3.1] for the three-dimensional case. For a detailed

explanation why the above result plays the role of a null structure we refer the reader to [1, Section 3].

4. Function Spaces

Based on the estimates developed in Section 2 we now define the function spaces in which we will perform the Picard iteration for (3.3). The construction here is a significant refinement of [1, Section 4]. Some of the similarities to the function spaces used in the wave map problem [10, 24, 26] are highlighted by using a similar notation.

For $1 \le p < \infty$ we define

$$\|f\|_{V^p_{\pm\langle D\rangle}} = \|f\|_{L^\infty_t L^2_x} + \Big(\sup_{(t_\nu)\in\mathcal{Z}} \sum_{\nu\in\mathbb{N}} \|e^{\mp it_{\nu+1}\langle D\rangle} f(t_{\nu+1}) - e^{\mp it_\nu \langle D\rangle} f(t_\nu)\|^p_{L^2_x} \Big)^{\frac{1}{p}},$$

where the supremum is taken over the set $\mathcal{Z}$ of all increasing sequences.

For the following, we consider a fixed $r \in \mathbb{N}$ (which is implicit in the definition, cf. Subsection 2.1).

For low frequencies, that is for $k \le 99$, we define

$$\|f\|_{S^\pm_k} = \|f\|_{V^2_{\pm\langle D\rangle}} + \sup_{\omega\in\mathbb{S}^1} \|f\|_{\sum_{\Lambda_{k,\omega}} L^2_{t_\Theta} L^\infty_{x_\Theta}}.$$

For the high frequencies, that is $k \ge 100$, the norm has a multiscale structure. We recall the notation convention that $\Lambda_{j,\kappa_1} = \Lambda_{j,\omega(\kappa_1)}$, and similarly for Ω_{j,κ_1}. Given $l \le k+10$, $\kappa \in \mathcal{K}_l$ and $j \ge 89$, we define structures $S^\pm[k,\kappa,j]$.

If $89 \le j = l-10 \le k-10$ or $l = k+10 \wedge j \ge k+10$, let

$$\|f\|_{S^\pm[k,\kappa,j]} = \sup_{\substack{\kappa_1\in\mathcal{K}_{j+10}:\\ \mathrm{d}(\kappa,\kappa_1)\le 2^{-l+3}}} \sup_{\Theta\in\Lambda_{j,\kappa_1}} 2^{-l} \|f\|_{L^\infty_{t^\pm_\Theta} L^2_{x^\pm_\Theta}}.$$

If $\max(90, l-9) \le \min(j,k) \le l+9$, let

$$\|f\|_{S^\pm[k,\kappa,j]} = \sup_{\substack{\kappa_1\in\mathcal{K}_{j+10}:\\ 2^{-l-3}\le\mathrm{d}(\kappa,\kappa_1)\le 2^{-l+3}}} \sup_{\Theta\in\Omega_{j,\kappa_1}} 2^{-\frac{l}{2}} \|f\|_{L^\infty_{x^{2,\pm}_\Theta} L^2_{(t,x^1)^\pm_\Theta}}$$

If $\max(90, l+10) \le \min(j,k)$, let

$$\|f\|_{S^\pm[k,\kappa,j]} = \sup_{\substack{\kappa_1\in\mathcal{K}_{j+10}:\\ 2^{-l-3}\le\mathrm{d}(\kappa,\kappa_1)\le 2^{-l+3}}} \sup_{\Theta\in\Lambda_{j,\kappa_1}} 2^{-l} \|f\|_{L^\infty_{t^\pm_\Theta} L^2_{x^\pm_\Theta}}$$

Then for $\kappa \in \mathcal{K}_l$ we define the cap localized structure as

$$\|f\|_{S^\pm[k,\kappa]} = \|f\|_{L^\infty_t L^2_x} + \sup_{\max(89,l-10)\le j} \|f\|_{S^\pm[k,\kappa,j]}.$$

We define the endpoint structure

$$\|f\|_{END^\pm_k} = \Big(\sum_{\kappa \in \mathcal{K}_{k+10}} 2^{-k} \|P_\kappa f\|^2_{\sum_{\Omega_{k,\kappa}} L^2_{x^2_\Theta} L^\infty_{(t,x^1)_\Theta}} + \|P_\kappa f\|^2_{\sum_{\Lambda_{k,\kappa}} L^2_{t^\pm_\Theta} L^\infty_{x^\pm_\Theta}} \Big)^{\frac{1}{2}}.$$

Next, for some $\frac{4}{3} < p < \frac{8}{5}$ (any p in this range will work, see Section 6) we define

$$\|f\|_{S^\pm_k} = \left(\sum_{\kappa \in \mathcal{K}_k} \|P_\kappa f\|^2_{V^2_{\pm\langle D \rangle}} \right)^{\frac{1}{2}} + 2^{(\frac{1}{p}-1)k} \sup_m 2^m \|Q^\pm_m f\|_{L^p_t L^2_x}$$
$$+ \|f\|_{END^\pm_k} + \sup_{1 \le l \le k+10} \Big(\sum_{\kappa \in \mathcal{K}_l} \|Q^\pm_{\prec k-2l} P_\kappa f\|^2_{S^\pm[k;\kappa]} \Big)^{\frac{1}{2}}$$

Remark 1. If $l_1 \geq l_2$, we have that for each $\kappa_1 \in \mathcal{K}_{l_1}$ the number of $\kappa_2 \in \mathcal{K}_{l_2}$ with $\kappa_1 \cap \kappa_2 \neq \emptyset$ is uniformly bounded. As a consequence, essential parts of this norm are square-summable with respect to caps: For later purposes, we note that for $l \le l'$,

$$\sum_{\kappa \in \mathcal{K}_l} \|P_\kappa f\|^2_{V^2_{\pm\langle D \rangle}} \lesssim \sum_{\kappa' \in \mathcal{K}_{l'}} \|P_{\kappa'} f\|^2_{V^2_{\pm\langle D \rangle}},$$

and, for all $1 \le l \prec k$,

$$\sum_{\kappa \in \mathcal{K}_l} \|P_\kappa f\|^2_{V^2_{\pm\langle D \rangle}} \lesssim \|f\|^2_{S^\pm_k}.$$

Similarly, we have

$$\sum_{\kappa' \in \mathcal{K}_l} \Big\{ \sum_{\kappa \in \mathcal{K}_{k+10}} 2^{-k} \|P_{\kappa'} P_\kappa f\|^2_{\sum_{\Omega_{k,\kappa}} L^2_{x^2_\Theta} L^\infty_{(t,x^1)_\Theta}} + \|P_{\kappa'} P_\kappa f\|^2_{\sum_{\Lambda_{k,\kappa}} L^2_{t^\pm_\Theta} L^\infty_{x^\pm_\Theta}} \Big\}$$
$$\lesssim \sum_{\kappa \in \mathcal{K}_{k+10}} 2^{-k} \|P_\kappa f\|^2_{\sum_{\Omega_{k,\kappa}} L^2_{x^2_\Theta} L^\infty_{(t,x^1)_\Theta}} + \|P_\kappa f\|^2_{\sum_{\Lambda_{k,\kappa}} L^2_{t^\pm_\Theta} L^\infty_{x^\pm_\Theta}} \lesssim \|f\|^2_{END^\pm_k}.$$

For this reason we introduce the norm

$$\|f\|_{l^2 S^\pm_k} = \left(\sum_{\kappa \in \mathcal{K}_k} \|P_\kappa f\|^2_{V^2_{\pm\langle D \rangle}} \right)^{\frac{1}{2}} + \|f\|_{END^\pm_k}$$

which has now the property that for any $1 \le l \le k+10$

$$\sum_{\kappa \in \mathcal{K}_l} \|P_\kappa f\|^2_{l^2 S^\pm_k} \lesssim \|f\|^2_{l^2 S^\pm_k}. \tag{4.1}$$

For any $|l - l'| \le 10$, we also have

$$\sum_{\kappa' \in \mathcal{K}_{l'}} \sum_{\kappa \in \mathcal{K}_l} \|P_{\kappa'} Q^\pm_{\prec k-2l} P_\kappa f\|^2_{S^\pm[k;\kappa]} \lesssim \|f\|^2_{S^\pm_k},$$

where we use Part i) of Lemma 4.1 below.

The space $S^{\pm,\sigma}$ corresponding to regularity at the level of $H^\sigma(\mathbb{R}^2)$ is the complete subspace of $L^\infty(\mathbb{R}, H^\sigma(\mathbb{R}^2))$ defined by the norm

$$\|f\|_{S^{\pm,\sigma}} = \|P_{\leq 89} f\|_{S^\pm_{89}} + \Big(\sum_{k \geq 90} 2^{2k\sigma} \|P_k f\|^2_{S^\pm_k} \Big)^{\frac{1}{2}}.$$

Recall from Subsection 2.1 that this construction is useful up to time 2^r, so for any closed interval $I \subset (-2^r, 2^r)$ we define the space $S^{\pm,\sigma}(I)$ of all functions on I which have extensions to functions in $S^{\pm,\sigma}$, with norm

$$\|f\|_{S^{\pm,\sigma}(I)} = \inf_{F \in S^{\pm,\sigma}} \{ \|F\|_{S^{\pm,\sigma}} : F|_I = f \}.$$

Note that the space $S^{\pm,\sigma}_C(I) := S^{\pm,\sigma}(I) \cap C(I, H^\sigma(\mathbb{R}^2))$ is a closed subspace of $S^{\pm,\sigma}(I)$.

Now we construct the space for the nonlinearity. For $1 \leq q \leq \infty$, $b \in \mathbb{R}$, we define

$$\|f\|_{\dot{X}^{\pm,b,q}} = \big\| \big(2^{bm} \|Q^\pm_m f\|_{L^2} \big)_{m \in \mathbb{Z}} \big\|_{\ell^q_m}.$$

For the low frequency part of the nonlinearity we define

$$\|f\|_{N^\pm_0} = \inf_{f = f_1 + f_2 + f_3} \Big\{ \|f_1\|_{\dot{X}^{\pm,-\frac{1}{2},1}} + \|f_2\|_{L^1_t L^2_x} + \|f_3\|_{L^{\frac{4}{3}}_{t,x}} \Big\} + \|f\|_{L^p_t L^2_x}.$$

Let $(N^\pm_0)^*$ denote the dual of $N^\pm_0$ and let $S^{\pm,w}_0$ be endowed with the norm

$$\|f\|_{S^{\pm,w}_0} = \|f\|_{L^\infty_t L^2_x} + \|f\|_{\dot{X}^{\pm,\frac{1}{2},\infty}}. \tag{4.2}$$

Then, we observe that for $k \leq 99$,

$$S^\pm_k \subset (N^\pm_0)^* \subset S^{\pm,w}_0. \tag{4.3}$$

Next, let $k \geq 100$. For $1 \leq l \leq k + 10$ we consider $\kappa \in \mathcal{K}_l$ and consider the following types of atoms:

A1 : If $89 \leq j = l - 10 \leq k - 10$ or $l = k + 10 \wedge j \geq k + 10$, functions f_Θ with

$$2^l \|f_\Theta\|_{L^1_{t^\pm_\Theta} L^2_{x^\pm_\Theta}} = 1,$$

where $\Theta \in \Lambda_{j,\kappa_1}$ and $\kappa_1 \in \mathcal{K}_{j+10}$ with $\mathrm{d}(\kappa_1, \kappa) \leq 2^{-l+3}$.

A2 : If $\max(90, l - 9) \leq \min(j, k) \leq l + 9$, functions f_Θ with

$$2^{\frac{l}{2}} \|f_\Theta\|_{L^1_{x^{2,\pm}_\Theta} L^2_{(t,x^1)^\pm_\Theta}} = 1,$$

where $\Theta \in \Omega_{j,\kappa_1}$ and $\kappa_1 \in \mathcal{K}_{j+10}$ with $2^{-l-3} \leq \mathrm{d}(\kappa_1, \kappa) \leq 2^{-l+3}$,

A3 : If $\max(90, l+10) \le j \le \min(j,k)$, functions f_Θ with

$$2^l \|f_\Theta\|_{L^1_{t^\pm_\Theta} L^2_{x^\pm_\Theta}} = 1,$$

where $\Theta \in \Lambda_{j,\kappa_1}$ and $\kappa_1 \in \mathcal{K}_{j+10}$ with $2^{-3} \le 2^l \,\mathrm{d}(\kappa_1, \kappa) \le 2^3$.

We then define, in the standard way, $N^\pm[k,\kappa]$ to be the atomic space based on the above atoms.

Now, similarly to [1], we define the following atomic structure

$$\begin{aligned}\|f\|_{N_k^{\pm,at}} = \inf_{f=f_1+f_2+\sum_{1\le l\le k+10} g_l} &\Big\{ \|f_1\|_{\dot{X}^{\pm,-\frac12,1}} + \|f_2\|_{L^1_t L^2_x} \\ &+ \sum_{1\le l\le k+10} \Big(\sum_{\kappa\in\mathcal{K}_l} \|P_\kappa g_l\|^2_{N^\pm[k,\kappa]}\Big)^{\frac12} \Big\}\end{aligned} \tag{4.4}$$

where the atoms g_l in the above decomposition are assumed to be localized at frequency 2^k and modulation $\ll 2^{k-2l}$, more precisely that $\tilde{Q}^\pm_{\prec k-2l} \tilde{P}_k g_l = g_l$.

The third component in $N_k^{\pm,at}$, i.e. the $\sum_{1\le l\le k+10} g_l$, will henceforth be called the cap-localized structure. The atoms g_l are localized in frequency and modulation, while when they are measured in $N^\pm[k,\kappa]$ the atoms a_Θ in the decomposition $g_l = \sum_\Theta a_\Theta$ are not assumed to keep that localization. However, by applying the operator $\tilde{Q}^\pm_{\prec k-2l}\tilde{P}_{k,\kappa}$ to the decomposition and using [1, Lemma 4.1 i)] (which holds true in dimension 2 verbatim) one obtains a new decomposition with similar norm. From now on our convention is that we assume that the atoms a_Θ in the atomic decomposition have the correct frequency and modulation localization.

Let $(N_k^{\pm,at})^*$ denote the dual of $N_k^{\pm,at}$ and $S_k^{\pm,w}$ be endowed with the norm

$$\|f\|_{S_k^{\pm,w}} = \|f\|_{L^\infty_t L^2_x} + \|f\|_{\dot{X}^{\pm,\frac12,\infty}} + \sup_{1\le l\le k+10} \Big(\sum_{\kappa\in\mathcal{K}_l} \|Q^\pm_{\prec k-2l} P_\kappa f\|^2_{S^\pm[k;\kappa]}\Big)^{\frac12}. \tag{4.5}$$

Then, we record that

$$S_k^\pm \subset (N_k^{\pm,at})^* \subset S_k^{\pm,w}, \tag{4.6}$$

with continuous embeddings, i.e.

$$\|f\|_{S_k^{\pm,w}} \lesssim \|f\|_{(N_k^{\pm,at})^*} \lesssim \|f\|_{S_k^\pm}.$$

Now we are in a position to define the space for dyadic pieces of the nonlinearity for high frequencies by setting

$$\|f\|_{N_k^\pm} = \|f\|_{N_k^{\pm,at}} + 2^{(\frac1p - 1)k} \|f\|_{L^p_t L^2_x}.$$

The space for the nonlinearity at regularity H^σ is defined via

$$\|f\|_{N^{\pm,\sigma}} = \|P_{\leq 89} f\|_{N^{\pm}_{\leq 89}} + \Big(\sum_{k \geq 90} 2^{2k\sigma} \|P_k f\|^2_{N^{\pm}_k} \Big)^{\frac{1}{2}}.$$

Now we show why the above structures are relevant for the equations we study. We first note a technical result on boundedness properties of certain frequency and modulation localization operators.

Lemma 4.1. i) *For all $k \geq 100$ and $m \geq 1$, the operators $\tilde{Q}^{\pm}_{\leq m}$ are bounded on $S^{\pm}_k, N^{\pm}_k$.*

ii) *For all $k \geq 100, 1 \leq l \leq k+10, \kappa \in \mathcal{K}_l$, and functions u localized at frequency 2^k, we have*

$$\| \left(\Pi_{\pm}(D) - \Pi_{\pm}(2^k \omega(\kappa))\right) P_\kappa u\|_S \lesssim 2^{-l} \|P_\kappa u\|_S \tag{4.7}$$

for $S \in \{S^{\pm}_k, S^{\pm,w}_k\}$.

Proof. i) We start with the boundedness of $\tilde{Q}^{\pm}_{\leq m}$ on the components of $S^{\pm}_k$. The boundedness of $\tilde{Q}^{\pm}_{\leq m}$ on the $V^2_{\pm\langle D\rangle}$ is standard, see e.g. [9, Cor. 2.18]. The boundedness of $\tilde{Q}^{\pm}_{\leq m}$ on the

$$2^{(\frac{1}{p}-1)k} \sup_{m'} 2^{m'} \|Q^{\pm}_{m'} f\|_{L^p_t L^2_x}$$

structure follows from the commutativity property $Q^{\pm}_{m'} \tilde{Q}^{\pm}_{\leq m} = \tilde{Q}^{\pm}_{\leq m} Q^{\pm}_{m'}$ and the boundedness of $\tilde{Q}^{\pm}_{\leq m}$ on the $L^p_t L^2_x$ type spaces.

Next, we notice that the kernel of $\tilde{Q}^{\pm}_{\leq m} \tilde{P}_\kappa$ belongs to $L^1_{t,x}$ under the hypothesis $m \geq 1$ and $\kappa \in \mathcal{K}_{k+10}$. Using that $P_\kappa \tilde{Q}^{\pm}_{\leq m} = \tilde{Q}^{\pm}_{\leq m} \tilde{P}_\kappa P_\kappa$, this implies the boundedness of $\tilde{Q}^{\pm}_{\leq m}$ on the

$$\Big(\sum_{\kappa \in \mathcal{K}_{k+10}} 2^{-k} \|P_\kappa f\|^2_{\sum_{\Omega_{k,\kappa}} L^2_{x^2_\Theta} L^\infty_{(t,x^1)_\Theta}} + \|P_\kappa f\|^2_{\sum_{\Lambda_{k,\kappa}} L^2_{t^{\pm}_\Theta} L^\infty_{x^{\pm}_\Theta}} \Big)^{\frac{1}{2}}$$

component of $S^{\pm}_k$.

For the boundedness of $\tilde{Q}^{\pm}_{\leq m}$ on the $S^{\pm}[k,\kappa]$ components we use an argument similar to the one used in [1, Lemma 4.1], part ii). $S^{\pm}[k,\kappa]$ itself has several components and we will provide a complete argument for one of them; this will also serve as a template for the other ones. With $\kappa \in \mathcal{K}_l$ for some $1 \leq l \leq k+10$, it is enough to consider only the case $m \prec k - 2l$. We fix the $+$ sign choice, fix j with $\max(90, l+10) \leq \min(j,k)$, consider κ_1 with $2^{-l-3} \leq \mathrm{d}(\kappa, \kappa_1) \leq 2^{-l+3}$ and $\Theta \in \Lambda_{j,\kappa_1}$.

The operator $\tilde{Q}^{+}_{\leq m} \tilde{P}_{k,\kappa}$ is a Fourier multiplier whose symbol

$$a_{m,k,\kappa}(\tau,\xi) = \tilde{\chi}_{\leq m}(\tau - \langle \xi \rangle) \tilde{\chi}_k(\xi) \tilde{\eta}_\kappa(\xi)$$

satisfies $|\partial^\beta_{\tau_\Theta} a_{m,k,\kappa}| \lesssim (2^{m+2l})^{-\beta}$. The inverse Fourier transform of $a_{m,k,\kappa}$ with respect to τ_{j,κ_1} satisfies

$$|K_{l,k,\kappa}(t_\Theta,\xi_\Theta)| \lesssim_N 2^{m+2l}(1+|t_\Theta|2^{m+2l})^{-N}, \text{ for any } N \in \mathbb{N}.$$

From this we obtain the uniform bound

$$\|K_{l,k,\kappa}\|_{L^1_{t_\Theta} L^\infty_{\xi_\Theta}} \lesssim 1.$$

On the other hand we have

$$\mathcal{F}_{\xi_\Theta}(\tilde{Q}^+_m \tilde{P}_{k,\kappa} f) = K_{l,k,\kappa} *_{t_\Theta} \mathcal{F}_{\xi_\Theta} f,$$

where one performs convolution with respect to t_Θ variable only. From the above statements it follows that $\tilde{Q}^+_{\leq m}\tilde{P}_{k,\kappa}$ is bounded on $L^\infty_{t_\Theta} L^2_{x_\Theta}$.

Proving the bounds for $\tilde{Q}^\pm_{\leq m}$ on the components of $N^\pm_k$ is done in an entirely similar way.

ii) The proof is similar to [1, Lemma 4.1] and therefore omitted. □

We continue with a few preparatory results. In order to later deal with the $V^2_{\pm\langle D\rangle}$ structure, we show that the analogue of the *fungibility estimate* [22, formula (159)] holds in our spaces, more precisely

Lemma 4.2. *For all $g = \tilde{P}_k g$ and any collection $(I_\nu)_{\nu\in\mathbb{N}}$ of disjoint intervals the estimate*

$$\sum_\nu \|1_{I_\nu} g\|^2_{N^\pm_k} \lesssim \|g\|^2_{N^\pm_k} \tag{4.8}$$

holds true, uniformly in $k \geq 100$.

Proof. We proceed similarly to [22, pp. 176-178], the minor differences in the following proof are mostly due to the lack of scale invariance:

It suffices to consider the $+$-case. It is obvious for $L^1_t L^2_x$-atoms, so we are left with $\dot{X}^{+,-\frac{1}{2},1}$-atoms and the cap-localized structure.

a) $\dot{X}^{+,-\frac{1}{2},1}$-atoms: We will prove

$$\sum_\nu \|1_{I_\nu} f_1\|^2_{L^1_t L^2_x + \dot{X}^{+,-\frac{1}{2},1}} \lesssim \|f_1\|^2_{\dot{X}^{+,-\frac{1}{2},1}}, \tag{4.9}$$

for $\tilde{P}_k f_1 = f_1$. By definition, this follows from

$$\sum_\nu \|1_{I_\nu} Q_m f_1\|^2_{L^1_t L^2_x + \dot{X}^{+,-\frac{1}{2},1}} \lesssim 2^{-m}\|Q_m f_1\|^2_{L^2}, \tag{4.10}$$

which we establish by proving

$$\sum_\nu \|Q_{\succeq m}(1_{I_\nu} Q_m f_1)\|^2_{L^2} \lesssim \|Q_m f_1\|^2_{L^2}, \tag{4.11}$$

$$\sum_\nu \|Q_{\prec m}(1_{I_\nu} Q_m f_1)\|^2_{L^1_t L^2_x} \lesssim 2^{-m}\|Q_m f_1\|^2_{L^2}. \tag{4.12}$$

The first one is trivial, since $Q_{\succeq m}$ is bounded in L^2, so we focus on (4.12): Let (J_ν) be the subcollection of all intervals in (I_ν) satisfying $|J_\nu| > 2^{-m}$ and (K_ν) all remaining intervals. For the short intervals (K_ν), we obtain

$$\sum_\nu \|Q_{\prec m}(1_{K_\nu} Q_m f_1)\|^2_{L^1_t L^2_x} \lesssim \sum_\nu \|1_{K_\nu} Q_m f_1\|^2_{L^1_t L^2_x} \lesssim 2^{-m} \sum_\nu \|1_{K_\nu} Q_m f_1\|^2_{L^2}.$$

Concerning the long intervals (J_ν), we have

$$Q_{\prec m}(1_{J_\nu} Q_m f_1) = Q_{\prec m}((Q_{\sim m} 1_{J_\nu})(Q_m f_1))$$

and it is easily checked that

$$|Q_{\sim m} 1_{[a,b]}(t)| \lesssim_N \alpha_{[a,b],m}(t)^{-N}, \quad \alpha_{[a,b],m}(t) := 1 + 2^m|t-a| + 2^m|t-b|.$$

Let $J_\nu = [a_\nu, b_\nu]$. Because of their disjointness and $|J_\nu| > 2^{-m}$, we have

$$\sum_\nu \alpha^{-N}_{[a_\nu,b_\nu],m}(t) \lesssim \sum_\nu (1 + 2^m|t-a_\nu| + 2^m|t-b_\nu|)^{-N} \lesssim 1 \quad (N > 1).$$

Fix $N = 2$. We conclude that

$$\begin{aligned} &\sum_\nu \|Q_{\prec m}(1_{J_\nu} Q_m f_1)\|^2_{L^1_t L^2_x} \lesssim \sum_\nu \|(Q_{\sim m} 1_{J_\nu})(Q_m f_1)\|^2_{L^1_t L^2_x} \\ \lesssim &\sum_\nu \|\alpha^{-1}_{[a_\nu,b_\nu],m}\|^2_{L^2_t} \|\alpha^{-1}_{[a_\nu,b_\nu],m} Q_m f_1\|^2_{L^2_t L^2_x} \\ \lesssim\, &2^{-m} \sum_\nu \|\alpha^{-1}_{[a_\nu,b_\nu],m}(t) Q_m f_1\|^2_{L^2_t L^2_x} \\ \lesssim\, &2^{-m} \int_{\mathbb{R}} \sum_\nu \alpha^{-2}_{[a_\nu,b_\nu],m}(t) \|Q_m f_1(t)\|^2_{L^2_x} dt \lesssim 2^{-m} \|Q_m f_1\|^2_{L^2}. \end{aligned}$$

b) cap-localized structure: Consider $f_3 = \sum_{1 \le l \le k+10} g_l$ satisfying $\tilde{Q}^+_{\prec k-2l} \tilde{P}_k g_l = g_l$. For fixed $1 \le l \le k+10$, we write

$$1_\nu g_l = \tilde{Q}^+_{\succeq k-2l}(1_\nu g_l) + \tilde{Q}^+_{\prec k-2l}(1_{I_\nu} g_l)$$

By a similar argument as presented in [1, Proof of Prop. 4.2, Part 1, Case c)] it follows that

$$\|P_\kappa g_l\|_{L^2_{t,x}} \lesssim 2^{\frac{k-2l}{2}} \|P_\kappa g_l\|_{N^+[k,\kappa]}.$$

For the first contribution, this implies

$$\begin{aligned} &\sum_\nu \|\tilde{Q}^+_{\succeq k-2l}(1_{I_\nu} g_l)\|^2_{\dot{X}^{+,-\frac{1}{2},1}} \lesssim 2^{2l-k} \sum_{\kappa \in \mathcal{K}_l} \sum_\nu \|(1_{I_\nu} P_\kappa g_l)\|^2_{L^2_{t,x}} \\ &\lesssim 2^{2l-k} \sum_{\kappa \in \mathcal{K}_l} \|P_\kappa g_l\|^2_{L^2_{t,x}} \lesssim \sum_{\kappa \in \mathcal{K}_l} \|P_\kappa g_l\|^2_{N^+[k,\kappa]}. \end{aligned}$$

For the second contribution we use Lemma 4.1 and the fact that

$$\sum_\nu \|1_{I_\nu} h\|^2_{L^1_{y_1}L^2_{y_2}} \lesssim \|h\|^2_{L^1_{y_1}L^2_{y_2}}$$

for any orthogonal frame $(y_1, y_2) \in \mathbb{R}^{1+2}$ due to Minkowski's inequality to deduce that for fixed $\kappa \in \mathcal{K}_l$ we have

$$\sum_\nu \|\tilde{Q}^+_{\prec k-2l}(1_{I_\nu} P_\kappa g_l)\|^2_{N^+[k,\kappa]} \lesssim \sum_\nu \|(1_{I_\nu} P_\kappa g_l)\|^2_{N^+[k,\kappa]} \lesssim \|P_\kappa g_l\|^2_{N^+[k,\kappa]},$$

which we then sum up with respect to $\kappa \in \mathcal{K}_l$. We obtain

$$\begin{aligned}
\Big(\sum_\nu \|1_{I_\nu} f_3\|^2_{N_k^{+,at}}\Big)^{\frac12} &\lesssim \sum_{1\le l\le k-10} \Big\{ \Big(\sum_\nu \|\tilde{Q}^+_{\succeq k-2l}(1_{I_\nu} g_l)\|^2_{\dot{X}^{+,-\frac12,1}}\Big)^{\frac12} \\
&+ \Big(\sum_\nu \sum_{\kappa\in\mathcal{K}_l} \|\tilde{Q}^+_{\prec k-2l}(1_{I_\nu} P_\kappa g_l)\|^2_{N^+[k,\kappa]}\Big)^{\frac12} \Big\} \\
&\lesssim \sum_{1\le l\le k+10} \Big(\sum_{\kappa\in\mathcal{K}_l} \|P_\kappa g_l\|^2_{N^+[k,\kappa]}\Big)^{\frac12},
\end{aligned}$$

and the proof is complete. □

Let ψ be any fixed Schwartz function and $\psi_T(\cdot) = \psi(\frac{\cdot}{T})$.

Lemma 4.3. *Fix any $1 \le p \le 2$. For all $T > 0$ we have*

$$\sup_{m\in Z} 2^m \|Q_m(\psi_T P_k f)\|_{L^p_t L^2_x} \lesssim \sup_{m\in\mathbb{Z}} 2^m \|Q_m P_k f\|_{L^p_t L^2_x} + T^{\frac1p - 1}\|P_k f\|_{L^\infty_t L^2_x}.$$

Consequently, there exists $c > 0$ such that for any closed interval $I \subset (-2^{r-1}, 2^{r-1})$, we have

$$\|e^{\pm it\langle D\rangle}\phi\|_{S^{\pm,\sigma}(I)} \le c\|\phi\|_{H^\sigma(\mathbb{R}^2)}. \tag{4.13}$$

Proof. Let $f = \tilde{P}_k f$. Obviously, $\|Q_{\lesssim m}\psi_T\|_{L^\infty_t} \lesssim 1$ and $[Q_m\psi_T](t) = [Q_{T2^m}\psi](\frac{t}{T})$, hence

$$\|Q_m\psi_T\|_{L^q_t} \lesssim_N T^{\frac1q}\langle T2^m\rangle^{-N} \text{ for any } N \in \mathbb{N}.$$

We split

$$Q_m(\psi_T f) = Q_m[Q_{\ll m}(\psi_T)f] + Q_m[Q_{\sim m}(\psi_T)f] + Q_m[Q_{\gg m}(\psi_T)f].$$

First,

$$2^m\|Q_m[Q_{\ll m}(\psi_T)f]\|_{L^p_t L^2_x} \lesssim \|Q_{\ll m}(\psi_T)\|_{L^\infty_t} 2^m \|Q_m f\|_{L^p_t L^2_x}.$$

Second,

$$\begin{aligned}
2^m\|Q_m[Q_{\sim m}(\psi_T)f]\|_{L^p_t L^2_x} &\lesssim \|Q_{\sim m}(\psi_T)\|_{L^p_t} 2^m \|f\|_{L^\infty_t L^2_x} \\
&\lesssim T^{\frac1p}\langle T2^m\rangle^{-1} 2^m \|f\|_{L^\infty_t L^2_x} \lesssim T^{\frac1p - 1}\|f\|_{L^\infty_t L^2_x}.
\end{aligned}$$

Third,

$$
\begin{aligned}
2^m\|Q_m[Q_{\gg m}(\psi_T)f]\|_{L^p_tL^2_x} &\lesssim 2^m\sum_{m_1\gg m}\|Q_{m_1}(\psi_T)Q_{m_1}f\|_{L^p_tL^2_x}\\
&\lesssim 2^m\sum_{m_1\gg m}\|Q_{m_1}(\psi_T)\|_{L^\infty_t}\|Q_{m_1}f\|_{L^p_tL^2_x}\\
&\lesssim \sum_{m_1\gg m}2^{m-m_1}\sup_{m_1}2^{m_1}\|Q_{m_1}f\|_{L^p_tL^2_x}.
\end{aligned}
$$

Concerning the second claim, we define the extension $F=\psi_Te^{\pm it\langle D\rangle}\phi$, where we choose ψ to be equal to 1 on $(-1,1)$, to be supported in $(-2,2)$ and ψ_T defined as above with $T=2^{r-1}$. The estimate follows from the first claim, the results from Section 2 and the fact that multiplication with smooth cutoffs is a bounded operation in V^2. □

Proposition 4.4. *i) For all $g\in N_k^\pm$ and initial data $u_0\in L^2(\mathbb{R}^2)$, both localized at (spatial) frequency 2^k (in the sense that $\tilde P_kg=g,\tilde P_ku_0=u_0$), $k\geq 100$, the solution u of*

$$(i\partial_t\pm\langle D\rangle)u=g,\quad u(0)=u_0, \tag{4.14}$$

satisfies $\psi_Tu\in S_k^\pm$ for all $1\lesssim T\lesssim 2^r$, and

$$\|\psi_Tu\|_{S_k^\pm}\lesssim\|g\|_{N_k^\pm}+\|u_0\|_{L^2}. \tag{4.15}$$

ii) A similar statement holds true for $90\leq k\leq 99$. For all $g\in N_{\leq 89}^\pm$ and initial data $u_0\in L^2(\mathbb{R}^2)$, both localized at (spatial) frequency $\leq 2^{89}$ (in the sense that $\tilde P_{\leq 89}g=g,\tilde P_{\leq 89}u_0=u_0$), the solution u of (4.14) *satisfies $\psi_Tu\in S_{\leq 89}^\pm$ for all $1\lesssim T\lesssim 2^r$, and*

$$\|\psi_Tu\|_{S_{\leq 89}^\pm}\lesssim\|g\|_{N_{\leq 89}^\pm}+\|u_0\|_{L^2}. \tag{4.16}$$

Proof. i) It suffices to consider the + case. Due to Lemma 4.3 it suffices to consider $u_0=0$. Our first claim is that we have the following estimate:

$$\|u\|_{S_k^+\setminus END_k^+}+\|\psi_Tu\|_{END_k^+}\lesssim\|g\|_{N_k^\pm} \tag{4.17}$$

where $S_k^+\setminus END_k^+$ contains all norm components of $S_k^\pm$ except the END_k^+ one. The time cut-off in is needed to recoup the END_k^+ structure. Besides the $V^2_{\langle D\rangle}$ component, the proof of (4.17) is analogous to the $3d$ case in [1, Prop. 4.2], which, in particular, implies the $L^\infty_tL^2_x$-bound. In what follows we provide the estimate for the $V^2_{\langle D\rangle}$ part of (4.17).

First, we follow the general strategy of [22, Prop. 5.4 and Lemma 5.8] to prove the $V^2_{\langle D\rangle}$-estimate on a fixed cap $\kappa\in\mathcal{K}_l$ with $l:=k+10$:

For any interval $[a,b]$ the function

$$w_\kappa(t) = P_\kappa u(t) - e^{i(t-a)\langle D\rangle} P_\kappa u(a)$$

solves

$$(i\partial_t \pm \langle D\rangle) w_\kappa = P_\kappa g, \quad w_\kappa(a) = 0,$$

hence we obtain, using the $L^\infty_t L^2_x$-bound,

$$\|P_\kappa u(b) - e^{i(b-a)\langle D\rangle} P_\kappa u(a)\|_{L^2_x} \lesssim \|1_{[a,b]} P_\kappa g\|_{N_k^+}.$$

For any $(t_\nu) \in \mathcal{Z}$, using (4.8), we conclude

$$\begin{aligned}\sum_\nu \|e^{-it_{\nu+1}\langle D\rangle} P_\kappa u(t_{\nu+1}) - e^{-it_\nu\langle D\rangle} P_\kappa u(t_\nu)\|_{L^2}^2 &\lesssim \sum_\nu \|1_{[t_\nu,t_{\nu+1}]} P_\kappa g\|_{N_k^+}^2\\ &\lesssim \|P_\kappa g\|_{N_k^+}^2,\end{aligned}$$

and finally we take the supremum over $\mathcal{Z}$.

Second, we sum up the squares: By the estimate above,

$$\Big(\sum_{\kappa\in\mathcal{K}_l} \|P_\kappa u\|^2_{V^2_{\langle D\rangle}}\Big)^{\frac12} \lesssim \Big(\sum_{\kappa\in\mathcal{K}_l} \|P_\kappa g\|^2_{N_k^+}\Big)^{\frac12},$$

hence it remains to prove

$$\Big(\sum_{\kappa\in\mathcal{K}_l} \|P_\kappa g\|^2_{N_k^+}\Big)^{\frac12} \lesssim \|g\|_{N_k^+}, \tag{4.18}$$

uniformly in $1 \le l \le k+10$. By Minkowski's inequality, this is obviously true for the $L^p_t L^2_x$-part of the N_k^+-norm, and also for the $\dot X^{+,-\frac12,1}$ and $L^1_t L^2_x$-atoms in $N_k^{+,at}$, so it remains to prove it for the cap-localized structure. We observe that

$$\begin{aligned}&\Big(\sum_{\kappa\in\mathcal{K}_{k+10}} \Big(\sum_{1\le l'\le k+10} \Big(\sum_{\kappa'\in\mathcal{K}_{l'}} \|P_{\kappa'} P_\kappa g\|^2_{N^+[k,\kappa']}\Big)^{\frac12}\Big)^2\Big)^{\frac12}\\ &\lesssim \sum_{1\le l'\le k+10} \Big(\sum_{\kappa'\in\mathcal{K}_{l'}} \sum_{\kappa\in\mathcal{K}_{k+10}} \|P_{\kappa'} P_\kappa g\|^2_{N^+[k,\kappa']}\Big)^{\frac12}.\end{aligned}$$

We now argue why (4.18) holds for the case when g is an atom in the cap localized structure. The only non-trivial case is when $g_\Theta = \tilde Q_{\prec k-2l'} \tilde P_{\kappa'} g_\Theta$ where $\kappa' \in \mathcal{K}_{l'}$ and $l' \le k+10$, while the information we have is control on $\|g_\Theta\|_{L^1_{t_\Theta} L^2_{x_\Theta}}$ or $\|g_\Theta\|_{L^1_{x^2_\Theta} L^2_{(t,x^1)_\Theta}}$ as described in A1 - A3 prior to the definition (4.4). Without restricting the generality of the argument, consider we have control of the first type. The key observation is that the operators $P_\kappa \tilde Q_{\prec k-2l'} \tilde P_{\kappa'}$ are almost orthogonal with respect to $\kappa \in \mathcal{K}_l$ when acting on $L^2_{x_\Theta}$. One way to formalize this is through the identity $P_\kappa \tilde Q_{\prec k-2l'} \tilde P_{\kappa'} = \tilde P(\kappa, \xi_\Theta) P_\kappa \tilde Q_{\prec k-2l'} \tilde P_{\kappa'}$ where

$\tilde{P}(\kappa, \xi_\Theta)$ are operators localizing the Fourier variable ξ_Θ in almost disjoint cap-type regions. This is a consequence of the transversality between the direction Θ and the Fourier support of $\tilde{Q}_{\prec k-2l'}\tilde{P}_{\kappa'}$.

Taking advantage of this almost orthogonality, we obtain

$$\sum_{\kappa \in \mathcal{K}_{k+10}} \|P_\kappa g_\Theta\|^2_{L^1_{t_\Theta} L^2_{x_\Theta}} \lesssim \|g_\Theta\|^2_{L^1_{t_\Theta} L^2_{x_\Theta}},$$

and this finishes the proof of (4.17).

Next we show how we derive (4.15) using (4.17). The problem encountered by a direct argument is that ψ_T does not commute well with the modulation localizations present in the $S^+[k,\kappa]$. $\psi_T u$ solves the following equation:

$$(i\partial_t \pm \langle D \rangle)(\psi_T u) = \psi_T g + i\psi_T' u. \tag{4.19}$$

with the initial data $\psi_T u(0) = u(0) = 0$. Since we have

$$\|i\psi_T' u\|_{L^1_t L^2_x} \lesssim \|u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{S^+_k} \lesssim \|g\|_{N_k}$$

and from the proof of Lemma 4.2 we easily obtain

$$\|\psi_T g\|_{N^+_k} \lesssim \|g\|_{N^+_k}. \tag{4.20}$$

We can invoke again (4.17), this time for the equation (4.19), to obtain

$$\|\psi_T u\|_{S^+_k \setminus END^+_k} \lesssim \|g\|_{N^+_k}.$$

This concludes the proof of (4.15).

ii) The proof of part ii) can be carried over in a similar but simpler way, except for the case when $g \in L^{\frac{4}{3}}_{t,x}$. A complete argument, including the $L^{\frac{4}{3}}_{t,x}$ part, can be found in [2, Proposition 7.2]. □

Corollary 4.5. *For any $r \in \mathbb{N}$, closed intervals $I \subset (-2^{r-1}, 2^{r-1})$, all $u_0 \in H^\sigma(\mathbb{R}^2)$ and $g \in N^{\pm,\sigma}$, there exists a unique solution $u \in S^{\pm,\sigma}(I)$ of* (4.14)*, and the following estimate holds true*

$$\|u\|_{S^{\pm,\sigma}(I)} \lesssim \|g\|_{N^{\pm,\sigma}(I)} + \|u_0\|_{H^\sigma}. \tag{4.21}$$

Proof. By definition of the spaces, it suffices to prove this for frequency localized functions which is provided by Proposition 4.4 above. □

Now, we conclude that we can control all non-endpoint Strichartz norms in our spaces, see also [10, 11, 24, 12] for other Strichartz type bounds. We refine the argument from [22] in the sense that we include additional cap-localizations which give stronger bounds.

Corollary 4.6. *Let $p, q \geq 2$ such that (p,q) is a Schrödinger-admissible pair, i.e.*

$$(p,q) \neq (2,\infty),\ \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \text{ and } s = 1 - \frac{2}{q}$$

or a wave admissible pair, i.e.

$$(p,q) \neq (4,\infty),\ \frac{2}{p} + \frac{1}{q} \leq \frac{1}{2}, \text{ and } s = 1 - \frac{2}{q} - \frac{1}{p}$$

i) *Then, we have*

$$\|P_k u\|_{L^p_t(\mathbb{R};L^q_x(\mathbb{R}^2))} \lesssim 2^{ks} \|P_k u\|_{S^\pm_k}. \tag{4.22}$$

ii) *Moreover, we have*

$$\sup_{1\leq l \leq k+10} \Big(\sum_{\kappa\in\mathcal{K}_l} \|P_k P_\kappa u\|^2_{L^p_t(\mathbb{R};L^q_x(\mathbb{R}^2))} \Big)^{\frac{1}{2}} \lesssim 2^{ks} \|P_k u\|_{S^\pm_k}. \tag{4.23}$$

Proof. It suffices to prove ii). The estimate holds for $P_k P_\kappa u$ in the atomic space $U^p_{\pm\langle D\rangle}$ because it is true for free solutions, which follows from TT^* argument and (2.11), hence it holds for U^p-atoms. Now, by changing $P_k P_\kappa u$ on a set of measure zero, we may assume that u is right-continuous, hence the claim follows from $\|P_k P_\kappa u\|_{U^p_{\pm\langle D\rangle}} \lesssim \|P_k P_\kappa u\|_{V^2_{\pm\langle D\rangle}}$, which holds for any $p > 2$, see [22, formula (189)], and [9, Section 2] for more details on these spaces. The claim follows from the definition of $\|\cdot\|_{S^\pm_k}$ and

$$\sup_{1\leq l\leq k+10} \sum_{\kappa\in\mathcal{K}_l} \|P_\kappa f\|^2_{V^2_{\pm\langle D\rangle}} \lesssim \sum_{\kappa\in\mathcal{K}_k} \|P_\kappa f\|^2_{V^2_{\pm\langle D\rangle}},$$

which is obvious. □

Clearly, one can also interpolate the estimates provided by Corollary 4.6 to obtain all Klein-Gordon admissible pairs (up to endpoints).

5. Bilinear and trilinear estimates

In this section we provide the crucial bilinear $L^2_{t,x}$-type estimates for functions in our spaces. For technical reasons, we also provide some trilinear estimates at the end of the section.

We use the same convention as in [1, Section 5] throughout the rest of the paper, namely that u's denote scalar-valued functions $u : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$, while ψ's denote vector-valued functions $\psi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}^2$. As before, a function f is said to be localized at frequency 2^k if $f = \tilde{P}_k f$ if $k \geq 90$ or $f = P_{\leq 90} f$ if $k = 89$. The first main result in this section is

Proposition 5.1. i) *For all* $k_1 \geq 89$ *and* $k_2 \geq 100$ *with* $10 \leq |k_1 - k_2|$ *and* $\psi_j \in S^{\pm}_{k_j}$ *localized at frequency* 2^{k_j} *for* $j = 1, 2$, *the following holds true:*

$$\left\| \langle \Pi_{\pm}(D)\psi_1, \beta \Pi_{\pm}(D)\psi_2 \rangle \right\|_{L^2} \lesssim 2^{\frac{k_1}{2}} \|\psi_1\|_{S^{\pm}_{k_1}} \|\psi_2\|_{S^{\pm,w}_{k_2}} \tag{5.1}$$

ii) *If in addition* $l \leq \min(k_1, k_2) + 10$, *then*

$$\left\| \sum_{\substack{\kappa_1,\kappa_2 \in \mathcal{K}_l: \\ \mathrm{d}(\pm\kappa_1, \pm\kappa_2) \lesssim 2^{-l}}} \langle \Pi_{\pm}(D)\tilde{P}_{\kappa_1}\psi_1, \beta \Pi_{\pm}(D)\tilde{P}_{\kappa_2}\psi_2 \rangle \right\|_{L^2} \lesssim 2^{\frac{k_1 - l}{2}} \|\psi_1\|_{S^{\pm}_{k_1}} \|\psi_2\|_{S^{\pm,w}_{k_2}}. \tag{5.2}$$

In both (5.1) *and* (5.2) *the sign of each* $\pm\kappa$ *and* $\Pi_{\pm}$ *is chosen to be consistent with the one of the corresponding* $S^{\pm}$.

iii) In the case $|k_1 - k_2| \leq 10$ *the above* (5.1)-(5.2) *hold true provided the following* parallel interaction *term is subtracted:*

$$\sum_{\substack{\kappa_1,\kappa_2 \in \mathcal{K}_{k_2}: \\ \mathrm{d}(\pm\kappa_1, \pm\kappa_2) \leq 2^{-k_2+3}}} \langle \Pi_{\pm}(D)\tilde{P}_{\kappa_1}\psi_1, \beta \Pi_{\pm}(D)\tilde{P}_{\kappa_2}\psi_2 \rangle.$$

iv) If $S^{\pm,w}_{k_2}$ *is replaced with* $S^{\pm}_{k_2}$, *then* (5.1) *and* (5.2) *improve as follows:*

- the factor becomes $2^{\frac{\min(k_1,k_2)}{2}}$, *respectively,* $2^{\frac{\min(k_1,k_2)-l}{2}}$;

- they hold for all $k_1, k_2 \geq 89$ *(in particular, no terms need to be subtracted in the case* $|k_1 - k_2| \leq 10$*).*

Proof of Proposition 5.1. To make the exposition easier, we choose to prove all the estimates for the $+$ choice in all terms. A careful examination of the argument reveals that the other choices follow in a similar manner.

We consider $k_1 \geq 89$ and $k_2 \geq 100$ and distinguish the following three cases: $k_1 \leq k_2 - 10, |k_1 - k_2| \leq 10$ and $k_1 \geq k_2 + 10$. We will work out in detail the first case, that is for $k_1 \leq k_2 - 10$. One should also note the close relation between these ranges and the ones given by the energy estimates in Theorem 2.3.

We will reduce (5.1) and (5.2) to the following claim: For all u_1, u_2 localized at frequencies 2^{k_1}, respectively 2^{k_2}, and $l \leq k_1 + 10$ the following estimate holds true:

$$\sum_{\kappa_1,\kappa_2 \in \mathcal{K}_l:*} \|\tilde{P}_{\kappa_1}u_1 \tilde{P}_{\kappa_2}u_2\|_{L^2} \lesssim 2^{\frac{k_1+l}{2}} \|u_1\|_{S^{+}_{k_1}} \|u_2\|_{S^{+,w}_{k_2}}, \tag{5.3}$$

where $*$ means that the above sum is restricted to the range $2^{-l-2} \leq \mathrm{d}(\kappa_1, \kappa_2) \leq 2^{-l+2}$ or $\mathrm{d}(\kappa_1, \kappa_2) \leq 2^{-l+2}$ in the case $l = k_1 + 10$.

We rely on the following estimate:

$$\sum_{\kappa_1,\kappa_2\in\mathcal{K}_l:*} \|\tilde{P}_{\kappa_1}u_1\cdot\tilde{P}_{\kappa_2}u_2\|_{L^2}\le A_1+A_2,$$

where

$$\begin{aligned}
A_1 &:= \sum_{\kappa_1,\kappa_2\in\mathcal{K}_l:*} \|\tilde{P}_{\kappa_1}u_1\|_{L^\infty}\|\tilde{P}_{\kappa_2}Q_{\succeq k_2-2l}u_2\|_{L^2}\\
&\lesssim 2^{\frac{2k_1-l}{2}}\Big(\sum_{\kappa_1\in\mathcal{K}_l}\|\tilde{P}_{\kappa_1}u_1\|^2_{L^\infty_t L^2_x}\Big)^{\frac12}\Big(\sum_{\kappa_2\in\mathcal{K}_l}\|\tilde{P}_{\kappa_2}Q_{\succeq k_2-2l}u_2\|^2_{L^2}\Big)^{\frac12}\\
&\lesssim 2^{\frac{2k_1-l}{2}}\Big(\sum_{\kappa_1\in\mathcal{K}_l}\|\tilde{P}_{\kappa_1}u_1\|_{L^\infty_t L^2_x}\Big)^{\frac12}2^{-\frac{k_2-2l}{2}}\|Q_{\succeq k_2-2l}u_2\|_{\dot{X}^{+,\frac12,\infty}}\\
&\lesssim 2^{\frac{k_1+l}{2}}\|u_1\|_{S^+_{k_1}}\|u_2\|_{S^{+,w}_{k_2}}.
\end{aligned}$$

The second term A_2, corresponding to the interaction $\tilde{P}_{\kappa_1}u_1\cdot Q_{\prec k_2-2l}\tilde{P}_{\kappa_2}u_2$, needs particular attention. We distinguish three particular scenarios $l\le k_1-11$, $k_1-10\le l\le k_1+9$ and $l=k_1+10$ and each of them is dealt with one of the three energy in frames components in the definition of $S^+[k_2,\kappa_2]$.

If $l\le k_1-11$, then we estimate as follows

$$\begin{aligned}
A_2 &:= \sum_{\kappa_1,\kappa_2\in\mathcal{K}_l:*}\sum_{\kappa\in\mathcal{K}_{k_1+10}}\|\tilde{P}_\kappa\tilde{P}_{\kappa_1}u_1\|_{\sum_{\Lambda_{k_1,\kappa}}L^2_{t_\Theta}L^\infty_{x_\Theta}}\\
&\qquad\cdot\sup_{\Theta\in\Lambda_{k_1,\kappa}}\|Q_{\prec k_2-2l}\tilde{P}_{\kappa_2}u_2\|_{L^\infty_{t_\Theta}L^2_{x_\Theta}}\\
&\lesssim\Big(\sum_{\kappa_2\in\mathcal{K}_l}\sup_{\substack{\kappa\in\mathcal{K}_{k_1+10}:\\ \kappa\cap\kappa_1\ne\emptyset}}\sup_{\Theta\in\Lambda_{k_1,\kappa}}\|Q_{\prec k_2-2l}\tilde{P}_{\kappa_2}u_2\|^2_{L^\infty_{t_\Theta}L^2_{x_\Theta}}\Big)^{\frac12}\\
&\qquad\cdot\Big(\sum_{\kappa_1\in\mathcal{K}_l}\Big(\sum_{\kappa\in\mathcal{K}_{k_1+10}}\|P_\kappa\tilde{P}_{\kappa_1}u_1\|_{\sum_{\Lambda_{k_1,\kappa}}L^2_{t_\Theta}L^\infty_{x_\Theta}}\Big)^2\Big)^{\frac12}\\
&\lesssim 2^{\frac{k_1-l}{2}}\|u_1\|_{S^+_{k_1}}2^l\|u_2\|_{S^{+,w}_{k_2}}.
\end{aligned}$$

If $k_1 - 10 \le l \le k_1 + 9$, then

$$\begin{aligned}
A_2 :=& \sum_{\kappa_1,\kappa_2 \in \mathcal{K}_l:*} \sum_{\kappa \in \mathcal{K}_{k_1+10}} \|P_\kappa \tilde{P}_{\kappa_1} u_1\|_{\sum_{\Omega_{k_1,\kappa}} L^2_{x^2_\Theta} L^\infty_{(t,x^1)_\Theta}} \\
&\cdot \sup_{\Theta \in \Omega_{k_1,\kappa}} \|Q_{\prec k_2 - 2l} \tilde{P}_{\kappa_2} u_2\|_{L^\infty_{t_\Theta} L^2_{x_\Theta}} \\
\lesssim& \Big(\sum_{\kappa_2 \in \mathcal{K}_l} \sup_{\substack{\kappa \in \mathcal{K}_{k_1+10}: \\ \kappa \cap \kappa_1 \neq \emptyset}} \sup_{\Theta \in \Omega_{k_1,\kappa}} \|Q_{\prec k_2 - 2l} \tilde{P}_{\kappa_2} u_2\|^2_{L^\infty_{t_\Theta} L^2_{x_\Theta}} \Big)^{\frac{1}{2}} \\
&\cdot \Big(\sum_{\kappa_1 \in \mathcal{K}_l} \Big(\sum_{\kappa \in \mathcal{K}_{k_1+10}} \|P_\kappa \tilde{P}_{\kappa_1} u_1\|_{\sum_{\Omega_{k_1,\kappa}} L^2_{t_\Theta} L^\infty_{x_\Theta}} \Big)^2 \Big)^{\frac{1}{2}} \\
\lesssim& 2^{\frac{k_1}{2}} \|\tilde{P}_{\kappa_1} u_1\|_{S^+_{k_1}} 2^{\frac{l}{2}} \|\tilde{P}_{\kappa_2} u_2\|_{S^{+,w}_{k_2}}.
\end{aligned}$$

If $l = k_1 + 10$, we repeat the argument of the first case without the additional localization to caps of size 2^{k_1+10}, and obtain

$$\begin{aligned}
A_2 :=& \sum_{\kappa_1,\kappa_2 \in \mathcal{K}_l:*} \|\tilde{P}_{\kappa_1} u_1\|_{\sum_{\Theta \in \Lambda_{k_1,\kappa_1}} L^2_{t_\Theta} L^\infty_{x_\Theta}} \sup_{\Theta \in \Lambda_{k_1,\kappa_1}} \|Q_{\prec k_2 - 2l} \tilde{P}_{\kappa_2} u_2\|_{L^\infty_{t_\Theta} L^2_{x_\Theta}} \\
\lesssim& \|u_1\|_{S^+_{k_1}} 2^{k_1} \|u_2\|_{S^{+,w}_{k_2}}.
\end{aligned}$$

Obviously, (5.3) implies

$$\sum_{\kappa_1,\kappa_2 \in \mathcal{K}_l:*} \|\tilde{P}_{\kappa_1} u_1 \overline{\tilde{P}_{\kappa_2} u_2}\|_{L^2} \lesssim 2^{\frac{k_1+l}{2}} \|u_1\|_{S^+_{k_1}} \|u_2\|_{S^{+,w}_{k_2}}. \tag{5.4}$$

Now, we turn to the proof of (5.2). Using (5.4) we claim the following

$$\begin{aligned}
&\sum_{\kappa_1,\kappa_2 \in \mathcal{K}_l:*} \|\langle \Pi_+(D) P_{\kappa_1} \psi_1, \beta \Pi_+(D) P_{\kappa_2} \psi_2 \rangle\|_{L^2} \\
&\lesssim 2^{\frac{k_1-l}{2}} \|\Pi_+(D)\psi_1\|_{S^+_{k_1}} \|\Pi_+(D)\psi_2\|_{S^{+,w}_{k_2}}.
\end{aligned} \tag{5.5}$$

To prove (5.5), we linearize the operator $\Pi_+(D)$ as follows

$$\Pi_+(D) = \Pi_+(2^{k_j}\omega(\kappa_j)) + \Pi_+(D) - \Pi_+(2^{k_j}\omega(\kappa_j))$$

where $j = 1, 2$. Taking into account (5.4) and (3.4) we obtain

$$\begin{aligned}
&\sum_{\kappa_1,\kappa_2 \in \mathcal{K}_l:*} \|\langle \Pi_+(2^{k_1}\omega(\kappa_1)) P_{\kappa_1} \psi_1, \beta \Pi_+(2^{k_2}\omega(\kappa_2)) P_{\kappa_2} \psi_2 \rangle\|_{L^2} \\
\lesssim& 2^{\frac{k_1-l}{2}} \|\psi_1\|_{S^+_{k_1}} \|\psi_2\|_{S^{+,w}_{k_2}}
\end{aligned}$$

where we have used $|\angle(\omega(\kappa_1), \omega(\kappa_2))| \lesssim 2^{-l}$ and that $\mathcal{O}(2^{-k_1} + 2^{-k_2}) \lesssim 2^{-k_1} \lesssim 2^{-l}$.

The estimate for the remaining terms follows from using (5.4) and (4.7). Now, we use

$$\begin{aligned} &\|\langle \Pi_+(D)\psi_1, \beta\Pi_+(D)\psi_2\rangle\|_{L^2} \\ \lesssim &\sum_{1\leq l\leq k_1+10}\ \sum_{\kappa_1,\kappa_2\in\mathcal{K}_l:*} \|\langle P_{\kappa_1}\Pi_+(D)\psi_1, \beta P_{\kappa_2}\Pi_+(D)\psi_2\rangle\|_{L^2}, \end{aligned}$$

and (5.5) and observe that the summation with respect to l is performed using the factor of $2^{-\frac{l}{2}}$.

This finishes the proof of i) and ii) in the case $k_1 \leq k_2 - 10$. The proof of (5.3) in the case $k_1 \geq k_2 + 10$ is similar in the case $l \leq k_2 - 11$ and $l = k_2 + 10$, and also in the case $k_2 - 10 \leq l \leq k_2 + 9$ for the contributions A_1. In the case of A_2, we modify the argument as in [1, Prop. 5.1]: We decompose

$$\tilde{P}_{\kappa_1}u_1 = \sum_{\kappa\in\mathcal{K}_{k_1+10}} P_\kappa \tilde{P}_{\kappa_1}u_1$$

and note that the interactions $P_\kappa \tilde{P}_{\kappa_1}u_1\tilde{P}_{\kappa_2}u_2$ are almost orthogonal with respect to $\kappa \in \mathcal{K}_{k_1+10}$, which follows from the fact that both $P_\kappa\tilde{P}_{\kappa_1}u_1$ and $\tilde{P}_{\kappa_2}u_2$ have Fourier-support of size ≈ 1 in the direction orthogonal to $\omega(\kappa_2)$. As a consequence

$$\|\tilde{P}_{\kappa_1}u_1\cdot\tilde{P}_{\kappa_2}Q_{\prec k_2-2l}u_2\|_{L^2}^2 \lesssim \sum_{\kappa\in\mathcal{K}_{k_1+10}} \|P_\kappa\tilde{P}_{\kappa_1}u_1\cdot\tilde{P}_{\kappa_2}Q_{\prec k_2-2l}u_2\|_{L^2}^2$$

and we can proceed as before.

The proof in the case $|k_1 - k_2| \leq 10$ is similar, except that there there is no mechanism to deal with the *parallel interactions*

$$\sum_{\substack{\kappa_1,\kappa_2\in\mathcal{K}_{k_2}:\\ \mathrm{d}(\kappa_1,\kappa_2)\leq 2^{-k_2+3}}} \langle \tilde{P}_{\kappa_1}u_1, \tilde{P}_{\kappa_2}u_2\rangle$$

in (5.3). This is the reason we cannot estimate this term and claim only the equivalent of (5.1)-(5.2) which excludes it.

Finally, the improvement in iv) is justified as follows: Since both terms are in S_k^+ type spaces, by symmetry reasons we can replace $2^{\frac{k_1}{2}}$ by $2^{\frac{\min(k_1,k_2)}{2}}$ in (5.1) and similarly in (5.2). If $89 \leq k_1, k_2 \leq 100$ we simply use the L^4-Strichartz bound on both functions. In the other cases where $|k_1 - k_2| \leq 10$ we use the fact that in S_k^+ we have access to the full family of Strichartz estimates for both terms and we estimate

the *parallel interactions* term as follows:

$$\begin{aligned}\Big\| \sum_{\substack{\kappa_1,\kappa_2\in\mathcal{K}_{k_2}:\\ \mathrm{d}(\kappa_1,\kappa_2)\leq 2^{-k_2+3}}} \langle \tilde{P}_{\kappa_1}u_1, \tilde{P}_{\kappa_2}u_2\rangle \Big\| &\lesssim \sum_{\substack{\kappa_1,\kappa_2\in\mathcal{K}_{k_2}:\\ \mathrm{d}(\kappa_1,\kappa_2)\leq 2^{-k_2+3}}} \|\tilde{P}_{\kappa_1}u_1\|_{L^4}\|\tilde{P}_{\kappa_2}u_2\|_{L^4}\\ &\lesssim \Big(\sum_{\kappa_1\in\mathcal{K}_{k_2}} \|\tilde{P}_{\kappa_1}u_1\|_{L^4}^2\Big)^{\frac12}\Big(\sum_{\kappa_2\in\mathcal{K}_{k_2}} \|\tilde{P}_{\kappa_2}u_2\|_{L^4}^2\Big)^{\frac12}\\ &\lesssim 2^{k_2}\|u_1\|_{S^+_{k_1}}\|u_2\|_{S^+_{k_2}}.\end{aligned}$$

This matches the numerology claimed in (5.3) and adds up correctly with the other angular interactions to give (5.1) and (5.2). ☐

Remark 2. The estimates of Proposition 5.1 can be interpolated with the trivial estimate

$$\||\psi_1||\psi_2|\|_{L^\infty_t L^2_x} \lesssim 2^{k_1}\|\psi_1\|_{L^\infty_t L^2_x}\|\psi_2\|_{L^\infty_t L^2_x}$$

obtain by the Bernstein inequality. In particular, for $2 \leq r \leq \infty$ we obtain

$$\big\|\langle \Pi_\pm(D)\psi_1, \beta\Pi_\pm(D)\psi_2\rangle\big\|_{L^r_t L^2_x} \lesssim 2^{k_1(1-\frac{1}{r})}\|\psi_1\|_{S^\pm_{k_1}}\|\psi_2\|_{S^\pm_{k_2}}. \tag{5.6}$$

We finish this section withtwo trilinear estimates.

Lemma 5.2. *Assume $k_1 \leq k_2 \leq k_3$ and each ψ_i is supported at frequency $2^{k_i}, i = 1,2,3$. The following estimate holds true for any $\frac{4}{3} < p \leq 2$ and any choice of signs $s_i \in \{\pm\}, i = 1,2,3$:*

$$\begin{aligned}&2^{(\frac{1}{p}-\frac{1}{2})k_3}\|\langle \Pi_{s_1}(D)\psi_1, \beta\Pi_{s_2}(D)\psi_2\rangle\beta\Pi_{s_3}(D)\psi_3\|_{L^p_t L^2_x}\\ &\lesssim 2^{(\frac{3}{8}-\frac{1}{2p})(k_1-k_2)}2^{(1-\frac{1}{p})(k_2-k_3)}\prod_{j=1}^{3}2^{\frac{k_j}{2}}\|\psi_j\|_{S^{s_j}_{k_j}}.\end{aligned} \tag{5.7}$$

Proof. The strategy is to recombine ψ_1 and ψ_3 or ψ_2 and ψ_3 and provide an L^2 type estimate as in (5.1). A careful analysis reveals that one can still extract gains from the null structure when recombining terms.

We provide a complete argument for the $\Pi_+(D)$ part of each term, that is we assume $\psi_i = \Pi_+(D)\psi_i, \forall i \in \{1,2,3\}$. A similar argument works for the other combinations. Fix $0 \leq l \leq k_1 + 10$ and write

$$I = \sum_{\kappa_1,\kappa_2\in\mathcal{K}_l:*} \langle \tilde{P}_{\kappa_1}\psi_1, \beta\tilde{P}_{\kappa_2}\psi_2\rangle\beta\psi_3$$

where $*$ indicates that we consider the range $2^{-l+3} \leq \mathrm{d}(\kappa_1,\kappa_2) \leq 2^{-l+6}$, if $l < k_1 + 10$, or $\mathrm{d}(\kappa_1,\kappa_2) \leq 2^{-l+6}$ in the case $l = k_1 + 10$.

Let $l < k_1 + 10$. Fix $\kappa_1, \kappa_2 \in \mathcal{K}_l$ subject to $*$. We explain now how to take advantage of the null condition in this context. For $j = 1, 2$ we decompose

$$\Pi_+(D) = \Pi_+(2^{k_j}\omega(\kappa_j)) + \Pi_+(D) - \Pi_+(2^{k_j}\omega(\kappa_j))$$

and use (3.4) and (4.7) to extract a factor of 2^{-l} from the expression $\langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2\rangle$ in all the computations below. To keep things simple in the estimates below, we skip the step where each $\psi_j, j = 1, 2$ goes through the above decomposition and simply just book the factor of 2^{-l}.

We start with the high modulation component of ψ_3 which we estimate as follows

$$\begin{aligned}
&\|\langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2\rangle \beta Q_{\succeq k_3-2l}\psi_3\|_{L^p_t L^2_x} \\
&\lesssim 2^{-l}\|\tilde{P}_{\kappa_1}\psi_1\|_{L^\infty_{t,x}}\|\tilde{P}_{\kappa_2}\psi_2\|_{L_t^{\frac{2p}{2-p}}L^\infty_x}\|Q_{\succeq k_3-2l}\psi_3\|_{L^2_{t,x}} \\
&\lesssim 2^{\frac{2k_1-l}{2}}\|\tilde{P}_{\kappa_1}\psi_1\|_{l^2 S^+_{k_1}} 2^{(1+\frac{p-2}{2p})k_2}\|\tilde{P}_{\kappa_2}\psi_2\|_{l^2 S^+_{k_2}} 2^{-\frac{k_3}{2}}\|\psi_3\|_{V^2_{+\langle D\rangle}}.
\end{aligned} \tag{5.8}$$

For the low modulation component, we decompose

$$\psi_3 = \sum_{l'<l-8}\sum_{\kappa_3\in\mathcal{K}^{**}_{l'}} P_{\kappa_3}\psi_3 + \sum_{\kappa_3\in\mathcal{K}^{***}_l} P_{\kappa_3}\psi_3 \tag{5.9}$$

where if $\kappa_3 \in \mathcal{K}^{**}_{l'}$, $d(\kappa_3, \kappa_1) \approx d(\kappa_3, \kappa_2) \approx 2^{-l'}$, while if $\kappa_3 \in \mathcal{K}^{***}_l$, $d(\kappa_3, \kappa_1) + d(\kappa_3, \kappa_2) \le 2^{-l+10}$. Fix $l' < l - 8$. Using (5.3) we estimate

$$\begin{aligned}
&\|\langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2\rangle \beta \sum_{\kappa_3\in\mathcal{K}^{**}_{l'}} P_{\kappa_3} Q_{\prec k_3-2l}\psi_3\|_{L^p_t L^2_x} \\
&\lesssim 2^{-l}\|\tilde{P}_{\kappa_1}\psi_1\|_{L_t^{\frac{2p}{2-p}}L^\infty_x} \cdot \Big\|\tilde{P}_{\kappa_2}\psi_2 \sum_{\kappa_3\in\mathcal{K}^{**}_{l'}} P_{\kappa_3} Q_{\prec k_3-2l}\psi_3\Big\|_{L^2} \\
&\lesssim 2^{-l}2^{(1+\frac{p-2}{2p})k_1}\|\tilde{P}_{\kappa_1}\psi_1\|_{l^2 S^+_{k_1}} 2^{\frac{k_2+l'}{2}}\|\tilde{P}_{\kappa_2}\psi_2\|_{l^2 S^+_{k_2}}\|\psi_3\|_{S^+_{k_3}},
\end{aligned}$$

since it follows from the proof of (5.3) that the operator $Q_{\prec k_3-2l}$ is disposable and we only need the $l^2 S_{k_2}$ component for $\tilde{P}_{\kappa_2}\psi_2$.

For the second sum, where $\kappa_3 \in \mathcal{K}^{***}_l$, the key property is that $2^{-l+1} \le \mathrm{d}(\kappa_3, \kappa_1) + \mathrm{d}(\kappa_3, \kappa_2) \lesssim 2^{-l}$. Thus we can split the set $\mathcal{K}^{***}_l = S_1 \cup S_2, S_1 \cap S_2 = \emptyset$ such that $\kappa_3 \in S_1$ satisfies $\mathrm{d}(\kappa_3, \kappa_1) \ge 2^{-l}$, while $\kappa_3 \in S_2$ satisfies $\mathrm{d}(\kappa_3, \kappa_2) \ge 2^{-l}$.

The part of the sum with $\kappa_3 \in S_2$ is estimated as above with $l' = l$, thus leading to

$$\begin{aligned}
&\|\langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2\rangle \beta \sum_{\kappa_3 \in S_2} P_{\kappa_3} Q_{\prec k_3 - 2l}\psi_3\|_{L_t^p L_x^2} \\
\lesssim & 2^{-l} 2^{(1+\frac{p-2}{2p})k_1} \|\tilde{P}_{\kappa_1}\psi_1\|_{l^2 S^+_{k_1}} 2^{\frac{k_2+l}{2}} \|\tilde{P}_{\kappa_2}\psi_2\|_{l^2 S^+_{k_2}} \|\psi_3\|_{S^+_{k_3}}.
\end{aligned}$$

The part of the sum with $\kappa_3 \in S_1$ is estimated as follows

$$\begin{aligned}
&\|\langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2\rangle \beta \sum_{\kappa_3 \in S_1} P_{\kappa_3} Q_{\prec k_3 - 2l}\psi_3\|_{L_t^p L_x^2} \\
\lesssim & 2^{-l} \|\tilde{P}_{\kappa_2}\psi_2\|_{L_t^{\frac{8p}{4-p}} L_x^\infty} \cdot \Big\| \tilde{P}_{\kappa_1}\psi_1 \sum_{\kappa_3 \in S_1} P_{\kappa_3} Q_{\prec k_3 - 2l}\psi_3 \Big\|_{L_t^{\frac{8p}{4+p}} L_x^2} \\
\lesssim & 2^{-l} 2^{(\frac{9}{8}-\frac{1}{2p})k_2} \|\tilde{P}_{\kappa_2}\psi_2\|_{l^2 S^+_{k_2}} 2^{(\frac{1}{p}+\frac{1}{4})\frac{k_1+l}{2}} 2^{(\frac{3}{4}-\frac{1}{p})k_1} \|\tilde{P}_{\kappa_1}\psi_1\|_{l^2 S^+_{k_1}} \|\psi_3\|_{S^+_{k_3}}.
\end{aligned}$$

The last inequality was obtained by interpolating between the two estimates

$$\Big\| \tilde{P}_{\kappa_1}\psi_1 \sum_{\kappa_3 \in S_1} P_{\kappa_3} Q_{\prec k_3 - 2l}\psi_3 \Big\|_{L_t^2 L_x^2} \lesssim 2^{\frac{k_1+l}{2}} \|\tilde{P}_{\kappa_1}\psi_1\|_{l^2 S^+_{k_1}} \|\psi_3\|_{S^+_{k_3}},$$

$$\Big\| \tilde{P}_{\kappa_1}\psi_1 \sum_{\kappa_3 \in S_1} P_{\kappa_3} Q_{\prec k_3 - 2l}\psi_3 \Big\|_{L_t^\infty L_x^2} \lesssim 2^{k_1} \|\tilde{P}_{\kappa_1}\psi_1\|_{l^2 S^+_{k_1}} \|\psi_3\|_{S^+_{k_3}},$$

where the first one follows from (5.3) and its proof, while the second one follows from the trivial estimate $\|\tilde{P}_{\kappa_1}\psi_1\|_{L^\infty_{t,x}} \lesssim 2^{k_1} \|\tilde{P}_{\kappa_1}\psi_1\|_{L_t^\infty L_x^2}$.

Bringing together the two inequalities we obtain:

$$\begin{aligned}
&\|\langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2\rangle \beta \sum_{\kappa_3 \in \mathcal{K}_l^{***}} P_{\kappa_3} Q_{\prec k_3 - 2l}\psi_3\|_{L_t^p L_x^2} \\
\lesssim & 2^{-\frac{l}{2}} 2^{(\frac{7}{8}-\frac{1}{2p})k_1} \|\tilde{P}_{\kappa_1}\psi_1\|_{l^2 S^+_{k_1}} 2^{(\frac{9}{8}-\frac{1}{2p})k_2} \|\tilde{P}_{\kappa_2}\psi_2\|_{l^2 S^+_{k_2}} \|\psi_3\|_{S^+_{k_3}},
\end{aligned}$$

At this time we can perform the summation with respect to the decomposition of ψ_3 in (5.9) to obtain:

$$\begin{aligned}
&\|\langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2\rangle \beta Q_{\prec k_3 - 2l}\psi_3\|_{L_t^p L_x^2} \\
\lesssim & 2^{-\frac{l}{2}} 2^{(\frac{7}{8}-\frac{1}{2p})k_1} \|\tilde{P}_{\kappa_1}\psi_1\|_{l^2 S^+_{k_1}} 2^{(\frac{9}{8}-\frac{1}{2p})k_2} \|\tilde{P}_{\kappa_2}\psi_2\|_{l^2 S^+_{k_2}} \|\psi_3\|_{S^+_{k_3}},
\end{aligned}$$

To this estimate we add the high modulation component estimate in (5.8) to conclude with

$$\begin{aligned}
&\|\langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2\rangle \beta \psi_3\|_{L_t^p L_x^2} \\
\lesssim & 2^{-\frac{l}{2}} 2^{(\frac{7}{8}-\frac{1}{2p})k_1} \|\tilde{P}_{\kappa_1}\psi_1\|_{l^2 S^+_{k_1}} 2^{(\frac{9}{8}-\frac{1}{2p})k_2} \|\tilde{P}_{\kappa_2}\psi_2\|_{l^2 S^+_{k_2}} \|\psi_3\|_{S^+_{k_3}},
\end{aligned}$$

The cap summation with respect to $\kappa_1, \kappa_2 \in \mathcal{K}_l : *$ is performed using the l^2 property of the $l^2 S_k$ spaces (4.1):

$$\begin{aligned}&\| \sum_{\kappa_1,\kappa_2 \in \mathcal{K}_l : *} \langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2\rangle \beta\psi_3\|_{L^p_t L^2_x} \\ \lesssim & 2^{-\frac{l}{2}} 2^{(\frac{7}{8}-\frac{1}{2p})k_1}\|\psi_1\|_{l^2 S^+_{k_1}} 2^{(\frac{9}{8}-\frac{1}{2p})k_2}\|\psi_2\|_{l^2 S^+_{k_2}}\|\psi_3\|_{S^+_{k_3}},\end{aligned}$$

Recall that up to this point we have used that $l < k_1+10$. If $l = k_1+10$, then one proceeds as above up to the point where we split the set $\mathcal{K}_l^{***} = S_1 \cup S_2$. The modification in this case is that we simply retain only the S_1 component which is now characterized by $d(\kappa_3, \kappa_1) \lesssim 2^{-k_1}$ and estimate as above to obtain

$$\begin{aligned}&\| \sum_{\kappa_1,\kappa_2 \in \mathcal{K}_l : *} \langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2\rangle \beta\psi_3\|_{L^p_t L^2_x} \\ \lesssim & 2^{-\frac{l}{2}} 2^{(\frac{7}{8}-\frac{1}{2p})k_1}\|\psi_1\|_{S^+_{k_1}} 2^{(\frac{9}{8}-\frac{1}{2p})k_2}\|\psi_2\|_{S^+_{k_2}}\|\psi_3\|_{S^+_{k_3}},\end{aligned}$$

where $l = k_1 + 10$. Finally, the summation with respect to l is done using the factor $2^{-\frac{l}{2}}$:

$$\begin{aligned}\|\langle\psi_1, \beta\psi_2\rangle\beta\psi_3\|_{L^p_t L^2_x} &\lesssim 2^{(\frac{7}{8}-\frac{1}{2p})k_1}\|\psi_1\|_{S_{k_1}} 2^{(\frac{9}{8}-\frac{1}{2p})k_2}\|\psi_2\|_{S_{k_2}}\|\psi_3\|_{S_{k_3}} \\ &\lesssim 2^{(\frac{3}{8}-\frac{1}{2p})k_1} 2^{(\frac{5}{8}-\frac{1}{2p})k_2} 2^{-\frac{k_3}{2}} \prod_{j=1}^{3} 2^{\frac{k_j}{2}}\|\psi_j\|_{S^+_{k_j}}\end{aligned}$$

from which (5.7) follows. □

Lemma 5.3. *Assume $k_1 \leq \min(k_2, k_3)$ and each ψ_i is supported at frequency $2^{k_i}, i = 1, 2, 3$. For any $2 \leq p \leq \infty$ and any choice of signs $s_i \in \{\pm\}, i = 1, 2, 3$, the following estimate holds true:*

$$\begin{aligned}&\|\Pi_{s_1}(D)\psi_1\langle\Pi_{s_2}(D)\psi_2, \beta\Pi_{s_3}(D)\psi_3\rangle\|_{L^p_t L^1_x} \\ \lesssim & 2^{(1-\frac{1}{p})k_1}\|\psi_1\|_{S^{s_1}_{k_1}}\|\psi_2\|_{S^{s_2}_{k_2}}\|\psi_3\|_{S^{s_3,w}_{k_3}}.\end{aligned} \tag{5.10}$$

Proof. Note that (5.10) follows from

$$\|\Pi_{s_1}(D)\psi_1\langle\Pi_{s_2}(D)\psi_2, \beta\Pi_{s_3}(D)\psi_3\rangle\|_{L^2_t L^1_x} \lesssim 2^{\frac{k_1}{2}}\|\psi_1\|_{S^{s_1}_{k_1}}\|\psi_2\|_{S^{s_2}_{k_2}}\|\psi_3\|_{S^{s_3,w}_{k_3}}, \tag{5.11}$$

by interpolating with the trivial estimate:

$$\begin{aligned}\|\psi_1\langle\psi_2, \beta\psi_3\rangle\|_{L^\infty_t L^1_x} &\lesssim \|\psi_1\|_{L^\infty_{t,x}}\|\psi_2\|_{L^\infty_t L^2_x}\|\psi_3\|_{L^\infty_t L^2_x} \\ &\lesssim 2^{k_1}\|\psi_1\|_{S^{s_1}_{k_1}}\|\psi_2\|_{S^{s_2}_{k_2}}\|\psi_3\|_{S^{s_3,w}_{k_3}}.\end{aligned}$$

Therefore the rest of this proof is concerned with (5.11). The argument carries some similarities with the one used in Lemma 5.2. In

particular we extract the gains from the null condition as explained in the body of that proof and skip the formalization here. We provide a complete argument for the $\Pi_+(D)$ part of each term, that is we assume $\psi_i = \Pi_+(D)\psi_i, \forall i \in \{1,2,3\}$. A similar argument works for the other combinations. We decompose
(5.12)

$$\psi_1\langle\psi_2, \beta\psi_3\rangle = \sum_{0\le l\le k+10} \sum_{\kappa_1\in\mathcal{K}_l} P_{\kappa_1}\psi_1 \sum_{i=2}^{3} \sum_{\kappa_2,\kappa_3\in\mathcal{K}_l^2(\kappa_1,i)} \langle P_{\kappa_2}\psi_2, \beta P_{\kappa_3}\psi_3\rangle$$

where $\mathcal{K}_l^2(\kappa_1,2) = \{(\kappa_2,\kappa_3) \in \mathcal{K}_l\times\mathcal{K}_l : 2^{-l+3} \le d(\kappa_1,\kappa_2) \le 2^{-l+6}, d(\kappa_1,\kappa_3) \le 2^{-l+6}\}$ and $\mathcal{K}_l^2(\kappa_1,3) = \{(\kappa_2,\kappa_3) \in \mathcal{K}_l \times \mathcal{K}_l : 2^{-l+3} \le d(\kappa_1,\kappa_3) \le 2^{-l+6}, d(\kappa_1,\kappa_2) \le 2^{-l+6}\}$ for $l < k_1 + 10$ while for $l = k_1 + 10$ we pick $\mathcal{K}_l^2(\kappa_1,2) = \mathcal{K}_l^2(\kappa_1,3) = \{(\kappa_2,\kappa_3) \in \mathcal{K}_l\times\mathcal{K}_l : d(\kappa_1,\kappa_3) \le 2^{-l+6}, d(\kappa_1,\kappa_2) \le 2^{-l+6}\}$. As defined, these sets are not disjoint, so we (implicitly) remove elements which are counted multiple times.

We fix $0 \le l < k_1 + 10$, $\kappa_1 \in \mathcal{K}_l$ and aim to estimate

$$\sum_{\kappa_1\in\mathcal{K}_l} P_{\kappa_1}\psi_1 \sum_{\kappa_2,\kappa_3\in\mathcal{K}_l^2(\kappa_1,2)} \langle P_{\kappa_2}\psi_2, \beta P_{\kappa_3}\psi_3\rangle$$

Notice that, given the structure of the set $\mathcal{K}_l^2(\kappa_1,2)$, for all $\kappa_2,\kappa_3 \in \mathcal{K}_l^2(\kappa_1,2)$ we have $d(\kappa_2,\kappa_3) \lesssim 2^{-l}$ and this allows us to book the gain of 2^{-l} from the null condition as explained in Lemma 5.2. Combining this with the fact that in the above sum we have $2^{-l+3} \le d(\kappa_1,\kappa_2) \le 2^{-l+6}$ we invoke (5.3) to obtain

$$\begin{aligned}
&\Big\| \sum_{\kappa_1\in\mathcal{K}_l} P_{\kappa_1}\psi_1 \sum_{\kappa_2,\kappa_3\in\mathcal{K}_l^2(\kappa_1,2)} \langle P_{\kappa_2}\psi_2, \beta P_{\kappa_3}\psi_3\rangle \Big\|_{L^2_t L^1_x} \\
\lesssim & 2^{-l} 2^{\frac{k_1+l}{2}} \|\psi_1\|_{S^+_{k_1}} \|\psi_2\|_{S^+_{k_2}} \sup_{\kappa_3} \|P_{\kappa_3}\psi_3\|_{L^\infty_t L^2_x} \\
\lesssim & 2^{\frac{k_1-l}{2}} \|\psi_1\|_{S^+_{k_1}} \|\psi_2\|_{S^+_{k_2}} \|\psi_3\|_{S^{+,w}_{k_3}}.
\end{aligned}$$

A similar argument gives

$$\Big\| \sum_{\kappa_1\in\mathcal{K}_l} P_{\kappa_1}\psi_1 \sum_{\kappa_2,\kappa_3\in\mathcal{K}_l^2(\kappa_1,3)} \langle P_{\kappa_2}\psi_2, \beta P_{\kappa_3}\psi_3\rangle \Big\|_{L^2_t L^1_x} \lesssim 2^{\frac{k_1-l}{2}} \|\psi_1\|_{S^+_{k_1}} \|\psi_2\|_{S^+_{k_2}} \|\psi_3\|_{S^{+,w}_{k_3}}.$$

If $l = k_1 + 10$ then we proceed as above in the case of $\mathcal{K}_l^2(\kappa_1,2)$ since ψ_2 comes with the stronger structure $S^+_{k_2}$.

To conclude with (5.11) we need to perform the summation with respect to l in (5.12); this is trivially done using the power of $2^{-\frac{l}{2}}$. $\square$

6. The Dirac nonlinearity

The main result of this section is the following

Theorem 6.1. *Choose $s_1, s_2, s_3, s_4 \in \{+, -\}$. Then, for all $\psi_k \in S^{s_k,\frac{1}{2}}$ satisfying $\psi_k = \Pi_{s_k}(D)\psi_k$ for $k = 1, 2, 3$, we have*

$$\|\Pi_{s_4}(D)(\langle\psi_1, \beta\psi_2\rangle\beta\psi_3)\|_{N^{s_4,\frac{1}{2}}} \lesssim \|\psi_1\|_{S^{s_1,\frac{1}{2}}}\|\psi_2\|_{S^{s_2,\frac{1}{2}}}\|\psi_3\|_{S^{s_3,\frac{1}{2}}}. \tag{6.1}$$

The rest of this section is devoted to the proof of Theorem 6.1 and the proof of our main result Theorem 1.1, which is organized similarly to [1, Section 6]. The estimate (6.1) will be derived from similar estimates for frequency localized functions. Our aim will be to identify a function $G(\mathbf{k}) : \mathbb{N}^4_{\geq 89} \to (0, \infty)$ such that

$$\sum_{k_1,k_2,k_3,k_4 \in \mathbb{N}_{\geq 89}} G(\mathbf{k}) a_{k_1} b_{k_2} c_{k_3} d_{k_4} \lesssim \|a\|_{l^2}\|b\|_{l^2}\|c\|_{l^2}\|d\|_{l^2} \tag{6.2}$$

for all sequences $a = (a_j)_{j\in\mathbb{N}_{\geq 89}}$, etc, in l^2. Here, we set $\mathbb{N}_{\geq 89} = \{n \in \mathbb{N} : n \geq 89\}$ and write $\mathbf{k} = (k_1, k_2, k_3, k_4)$.

With these notations, the result of Theorem 6.1 follows from

Proposition 6.2. *There exists a function G satisfying* (6.2) *such that if ψ_j are localized at frequency 2^{k_j}, $k_j \geq 89$ and $\psi_j = \Pi_{s_j}(D)\psi_j$ for $j = 1, 2, 3$, then the following holds true*

$$2^{\frac{k_4}{2}}\|P_{k_4}\Pi_{s_4}(D)(\langle\psi_1, \beta\psi_2\rangle\beta\psi_3)\|_{N^{s_4}_{k_4}} \lesssim G(\mathbf{k})\prod_{j=1}^{3} 2^{\frac{k_j}{2}}\|\psi_j\|_{S^{s_j}_{k_j}}, \tag{6.3}$$

for any choice of sign $s_1, s_2, s_3, s_4 \in \{+, -\}$.

We break this down into two building blocks:

Lemma 6.3. *Under the assumptions of Proposition 6.2 the following estimate holds true for any $\frac{4}{3} < p \leq 2$:*

$$2^{(\frac{1}{p}-\frac{1}{2})k_4}\|P_{k_4}\Pi_{s_4}(D)(\langle\psi_1, \beta\psi_2\rangle\beta\psi_3)\|_{L^p_t L^2_x} \lesssim G(\mathbf{k})\prod_{j=1}^{3} 2^{\frac{k_j}{2}}\|\psi_j\|_{S^{s_j}_{k_j}}. \tag{6.4}$$

Lemma 6.4. *Under the assumptions of Proposition 6.2 (including now that ψ_4 are localized at frequency 2^{k_4} and $\psi_4 = \Pi_{s_4}(D)\psi_4$) the following estimate hold true:*

$$\Big|\int \langle\psi_1, \beta\psi_2\rangle \cdot \langle\psi_3, \beta\psi_4\rangle dxdt\Big| \lesssim G(\mathbf{k})\prod_{j=1}^{3} 2^{\frac{k_j}{2}}\|\psi_j\|_{S^{s_j}_{k_j}} \cdot 2^{-\frac{k_4}{2}}\|\psi_4\|_{S^{s_4,w}_{k_4}}. \tag{6.5}$$

Next, we show how Lemmas 6.3 and 6.4 imply Proposition 6.2.

Proof of Prop. 6.2. The estimate (6.4) provides the $L_t^p L_x^2$ part of (6.3). Next, we explain why (6.5) implies the atomic part of (6.3). The nonlinearity

$$\mathcal{N} = P_{k_4}\Pi_{s_4}(D)(\langle\psi_1, \beta\psi_2\rangle\beta\psi_3)$$

satisfies $\mathcal{N} = \tilde{P}_{k_4}\Pi_{s_4}(D)\mathcal{N}$ and has to be estimated in $N_{k_4}^{s_4}$. Using the duality (4.6), it suffices to test $\mathcal{N}$ against $\psi_4 \in S_{k_4}^{s_4,w}$ and to prove the estimate

(6.6)

$$\Big|\int \langle P_{k_4}\Pi_{s_4}(D)\mathcal{N}, \psi_4\rangle dxdt\Big| \lesssim G(\mathbf{k}) \prod_{j=1}^{3} 2^{\frac{k_j}{2}} \|\psi_j\|_{S_{k_j}^{s_j}} \cdot 2^{-\frac{k_4}{2}} \|\psi_4\|_{S_{k_4}^{s_4,w}}.$$

We have

$$\begin{aligned}\int \langle \mathcal{N}, \psi_4\rangle dxdt &= \int \langle\langle\psi_1, \beta\psi_2\rangle\beta\psi_3, \Pi_{s_4}(D)P_{k_4}\psi_4\rangle dxdt\\ &= \int \langle\psi_1, \beta\psi_2\rangle\langle\psi_3, \beta\Pi_{s_4}(D)P_{k_4}\psi_4\rangle dxdt.\end{aligned}$$

Now, we split $\psi_j = \Pi_+(D)\psi_j + \Pi_-(D)\psi_j$, and each contribution to (6.6) is bounded by (6.5). $\square$

Proof of Lemma 6.3. We will use the notation:

$$TR = 2^{\frac{k_1}{2}}\|\psi_1\|_{S_{k_1}^{s_1}} 2^{\frac{k_2}{2}}\|\psi_2\|_{S_{k_2}^{s_2}} 2^{\frac{k_3}{2}}\|\psi_3\|_{S_{k_3}^{s_3}}.$$

The argument is symmetric with respect to k_1, k_2, hence we can simply assume that $k_1 \leq k_2$.

We first consider the case $k_3 \leq k_1 + 20$, in which case $k_4 \leq k_2 + 30$ or else the l.h.s. of (6.4) vanishes. Using Strichartz and Prop. 5.1, we obtain

$$\begin{aligned}\|\langle\psi_1, \beta\psi_2\rangle\beta\psi_3\|_{L_t^p L_x^2} &\lesssim \|\langle\psi_1, \beta\psi_2\rangle\|_{L^2}\|\psi_3\|_{L_t^{\frac{2p}{2-p}} L_x^\infty}\\ &\lesssim 2^{\frac{k_1}{2}}\|\psi_1\|_{S_{k_1}^{s_1}}\|\psi_2\|_{S_{k_2}^{s_2}} 2^{(1+\frac{p-2}{2p})k_3}\|\psi_3\|_{S_{k_3}^{s_3}}\\ &\lesssim 2^{\frac{p-2}{2p}k_4} 2^{\frac{k_3-k_2}{2}} 2^{\frac{p-2}{2p}(k_3-k_4)} TR\end{aligned}$$

which is acceptable given that $0 \leq \frac{2-p}{2p} < \frac{1}{2}$.

If $k_1 + 20 \leq k_3 \leq k_2 + 20$ we use (5.6) and obtain

$$\begin{aligned}\|\langle\psi_1, \beta\psi_2\rangle\beta\psi_3\|_{L_t^p L_x^2} &\lesssim \|\langle\psi_1, \beta\psi_2\rangle\|_{L_t^{\frac{8p}{p+4}} L_x^2}\|\psi_3\|_{L_t^{\frac{8p}{4-p}} L_x^\infty}\\ &\lesssim 2^{(\frac{7}{8}-\frac{1}{2p})k_1}\|\psi_1\|_{S_{k_1}^{s_1}}\|\psi_2\|_{S_{k_2}^{s_2}} 2^{(\frac{9}{8}-\frac{1}{2p})k_3}\|\psi_3\|_{S_{k_3}^{s_3}}\\ &\lesssim 2^{\frac{p-2}{2p}k_4} 2^{\frac{2-p}{2p}(k_4-k_2)} 2^{(\frac{3}{8}-\frac{1}{2p})(k_1-k_2)} 2^{(\frac{5}{8}-\frac{1}{2p})(k_3-k_2)} TR\end{aligned}$$

which is acceptable given that $\frac{4}{3} < p \leq 2$.

Next we consider the case $k_2 + 20 \leq k_3$, in which case $k_4 \leq k_3 + 10$ or else the l.h.s. of (6.4) vanishes. In this case the estimate (5.7) gives the desired bound provided that $\frac{4}{3} < p \leq 2$. $\square$

It remains to prove Lemma 6.4. Before we start to do so, we analyze the modulation of a product of two waves as in [1]. We consider two functions $\psi_1, \psi_2 \in S^+$ where their native modulation is with respect to the quantity $|\tau - \langle \xi \rangle|$. However, for $\langle \psi_1, \beta\psi_2 \rangle$ we quantify the output modulation with respect to $||\tau| - \langle \xi \rangle|$. We recall from [1] the following lemma which contains the modulation localization claim which will be used several times in the argument.

Lemma 6.5. *i) Let $k, k_1 k_2 \geq 100$ and $l \prec \min(k_1, k_2)$, and let $\kappa_1, \kappa_2 \in \mathcal{K}_l$, with $\mathrm{d}(\kappa_1, \kappa_2) \approx 2^{-l}$, and assume that $u_j = \tilde{P}_{k_j,\kappa_j} \tilde{Q}^{s_j}_{\prec m} u_j$, where*

$$m = k_1 + k_2 - k - 2l.$$

Then, if $s_1 = s_2$,

$$\widehat{P_k(u_1 \overline{u_2})}(\tau, \xi) = 0 \text{ unless } ||\tau| - \langle \xi \rangle| \approx 2^m.$$

ii) Using the same setup as in part i) but with $s_1 = -s_2$ and $\mathrm{d}(\kappa_1, -\kappa_2) \approx 2^{-l}$, the same result applies with

$$m = \min(k_1, k_2) - 2l.$$

Proof. i) The proof of the same result in [1] (where we worked in dimension 3) does not involve the dimension of the physical space, thus it carries over verbatim to dimension 2 for $s_1 = s_2 = +$. The argument $s_1 = s_2 = -$ is entirely similar.

ii) Since the modulation of the inputs are much less than the claimed modulation of the output it is enough to prove the argument for free solutions. Let $(\xi_1, \langle \xi_1 \rangle)$ be in the support of $\hat{u}_1$ and $(-\xi_2, \langle \xi_2 \rangle)$ be in the support of $\overline{\hat{u}_2}$. Then, the angle between ξ_1 and ξ_2 is $\approx 2^{-l}$. Let $\xi = \xi_1 - \xi_2$ be of size 2^k and $\tau = \langle \xi_1 \rangle - \langle \xi_2 \rangle$. Our aim is to prove that

$$|\langle \xi_1 - \xi_2 \rangle - |\langle \xi_1 \rangle + \langle \xi_2 \rangle|| \approx 2^m.$$

The claim follows from

$$\begin{aligned} \langle \xi_1 - \xi_2 \rangle - |\langle \xi_1 \rangle + \langle \xi_2 \rangle| &= \frac{\langle \xi_1 - \xi_2 \rangle^2 - (\langle \xi_1 \rangle + \langle \xi_2 \rangle)^2}{\langle \xi_1 - \xi_2 \rangle + |\langle \xi_1 \rangle + \langle \xi_2 \rangle|} \\ &= \frac{2|\xi_1||\xi_2|(1 + \cos(\angle(\xi_1, \xi_2)))}{\langle \xi_1 - \xi_2 \rangle + |\langle \xi_1 \rangle + \langle \xi_2 \rangle|} + \mathcal{O}(2^{-\max(k_1,k_2)}) \\ &\approx 2^{\min(k_1,k_2)} \angle(\xi_1, \xi_2)^2 \end{aligned}$$

because by assumption we have $2^{\min(k_1,k_2)-2l} \gg 2^{-\max(k_1,k_2)}$. $\square$

Proof of Lemma 6.4. Without restricting the generality of the argument we prove (6.5) for the + choice in all terms. Once we finish the argument for the + choice in all terms, we indicate how the other cases are treated. Thus, for now, we drop all the $\pm$ and simply consider $\psi_j \in S^+_{k_j}$ and write $S_{k_j} = S^+_{k_j}$ instead.

For brevity, we denote the l.h.s. of (6.5) as

$$I := \Big| \int \langle \psi_1, \beta\psi_2 \rangle \cdot \langle \psi_3, \beta\psi_4 \rangle dxdt \Big|$$

and the standard factor on the r.h.s. as

$$J := \prod_{j=1}^{3} 2^{\frac{k_j}{2}} \|\psi_j\|_{S_{k_j}} \cdot 2^{-\frac{k_4}{2}} \|\psi_4\|_{S^w_{k_4}}.$$

Since the expression I computes the zero mode of the product $\langle \psi_1, \beta\psi_2 \rangle \cdot \langle \psi_3, \beta\psi_4 \rangle$, it follows that $\langle \psi_1, \beta\psi_2 \rangle$ and $\langle \psi_3, \beta\psi_4 \rangle$ need to be localized at frequencies and modulations of comparable size, where the modulation is computed with respect to $||\tau| - \langle \xi \rangle|$. This will be repeatedly used in the argument below along with the convention that the modulations of $\psi_k, k = 1, \dots, 4$ are with respect to $|\tau - \langle \xi \rangle|$, while the modulations of $\langle \psi_1, \beta\psi_2 \rangle$ and $\langle \psi_3, \beta\psi_4 \rangle$ are with respect to $||\tau| - \langle \xi \rangle|$.

We also agree that by the angle of interaction in, say, $\langle \psi_1, \beta\psi_2 \rangle$ we mean the angle made by the frequencies in the support of $\hat{\psi}_1$ and $\hat{\psi}_2$, where we consider only the supports that bring nontrivial contributions to I.

We organize the argument based on the size of the frequencies. There are a two easy cases we can easily dispose of.

<u>Case 1: $\max(k_1, k_2, k_3, k_4) \leq 200$.</u> In this case we estimate

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^3_t L^6_x} \|\psi_2\|_{L^3_t L^6_x} \|\psi_3\|_{L^3_t L^6_x} \|\psi_4\|_{L^\infty_t L^2_x} \\ &\lesssim \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim J \end{aligned}$$

<u>Case 2: $k_4 < 100$.</u> Using (5.1) in the context of part iv) of Proposition 5.1 we obtain:

$$\begin{aligned} I &\lesssim \|\langle \psi_1, \beta\psi_2 \rangle\|_{L^2} \|\psi_3\|_{L^4} \|\psi_4\|_{L^4} \\ &\lesssim 2^{\frac{\min(k_1,k_2)}{2}} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{\frac{k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{-\frac{\max(k_1,k_2)}{2}} J \end{aligned}$$

Given that, in order to account for nontrivial outputs, we need to consider only the case when $k_3 \prec \max(k_1, k_2)$, the above estimate suffices.

We continue with the more delicate cases. In light of Case 2, from now on we work under the hypothesis that $k_4 \geq 100$.

Case 3: $k_4 \leq \min(k_1, k_2, k_3) + 10$. If $k_3 \geq k_4 + 10$, then we use (5.1) and (5.2) to obtain

$$|I| \lesssim 2^{\frac{\min(k_1,k_2)}{2}} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{\frac{k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \lesssim 2^{\frac{k_4 - \max(k_1,k_2)}{2}} J.$$

which is acceptable given that $k_3 \leq \max(k_1, k_2) + 10$ (or else $I = 0$).

If $k_4 - 10 \leq k_3 \leq k_4 + 9$ the above argument covers most of I except

$$I_{par} := \Big| \sum_{\substack{\kappa_3,\kappa_4 \in \mathcal{K}_{k_4}: \\ \mathrm{d}(\kappa_3,\kappa_4) \leq 2^{-k_4+3}}} \int \langle \psi_1, \beta\psi_2 \rangle \cdot \langle \tilde{P}_{\kappa_3}\psi_3, \beta \tilde{P}_{\kappa_4}\psi_4 \rangle dxdt \Big|$$

If $k_1, k_2 \leq k_4 + 15$ this is estimated as follows:

$$\begin{aligned} I_{par} &\lesssim \|\psi_1\|_{L^3_t L^6_x} \|\psi_2\|_{L^3_t L^6_x} 2^{-k_4} \sum_{\substack{\kappa_3,\kappa_4 \in \mathcal{K}_{k_4}: \\ \mathrm{d}(\kappa_3,\kappa_4) \leq 2^{-k_4+3}}} \|\tilde{P}_{\kappa_3}\psi_3\|_{L^3_t L^6_x} \|\tilde{P}_{\kappa_4}\psi_4\|_{L^\infty_t L^2_x} \\ &\lesssim 2^{\frac{2(k_1+k_2)}{3} - k_4} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} \left(\sum_{\kappa_3 \in \mathcal{K}_{k_4}} \|\tilde{P}_{\kappa_3}\psi_3\|^2_{L^3_t L^6_x} \right)^{\frac{1}{2}} \left(\sum_{\kappa_4 \in \mathcal{K}_{k_4}} \|\tilde{P}_{\kappa_4}\psi_4\|^2_{L^\infty_t L^2_x} \right)^{\frac{1}{2}} \\ &\lesssim 2^{\frac{2(k_1+k_2+k_3)}{3} - k_4} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim J \end{aligned}$$

where we have used that $|k_i - k_4| \leq 15$, for $i \in \{1, 2, 3\}$.

If $k_1 \geq k_4 + 15$, then $k_2 \geq k_4 + 10$. In addition, since $\langle \psi_1, \beta\psi_2 \rangle$ is supported at frequency $\lesssim 2^{k_4}$, it follows that only the interactions between ψ_1 and ψ_2 making an angle $\lesssim 2^{k_4 - k_1}$ have nontrivial contribution to I. Therefore we need to consider only

$$I_{par} := \Big| \sum_{\substack{\kappa_1,\kappa_2 \in \mathcal{K}_{k_1-k_4}: \\ \mathrm{d}(\kappa_1,\kappa_2) \lesssim 2^{k_4-k_1}}} \sum_{\substack{\kappa_3,\kappa_4 \in \mathcal{K}_{k_4}: \\ \mathrm{d}(\kappa_3,\kappa_4) \leq 2^{-k_4+3}}} \int \langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2 \rangle \cdot \langle \tilde{P}_{\kappa_3}\psi_3, \beta \tilde{P}_{\kappa_4}\tilde{\psi}_4 \rangle dxdt \Big|$$

Now we use a similar argument to the one when $k_1, k_2 \leq k_4 + 15$:

$$
\begin{aligned}
I_{par} \lesssim 2^{k_4-k_1} & \sum_{\substack{\kappa_1,\kappa_2 \in \mathcal{K}_{k_1-k_4}: \\ \mathrm{d}(\kappa_1,\kappa_2) \lesssim 2^{k_4-k_1}}} \|\tilde{P}_{\kappa_1}\psi_1\|_{L_t^3 L_x^6} \|\tilde{P}_{\kappa_2}\psi_2\|_{L_t^3 L_x^6} \\
& \cdot 2^{-k_4} \sum_{\substack{\kappa_3,\kappa_4 \in \mathcal{K}_{k_4}: \\ \mathrm{d}(\kappa_3,\kappa_4) \leq 2^{-k_4+3}}} \|\tilde{P}_{\kappa_3}\psi_3\|_{L_t^3 L_x^6} \|\tilde{P}_{\kappa_4}\psi_4\|_{L_t^\infty L_x^2} \\
& \lesssim 2^{-k_1} \left(\sum_{\kappa_1 \in \mathcal{K}_{k_1-k_4}} \|\tilde{P}_{\kappa_1}\psi_1\|^2_{L_t^3 L_x^6} \right)^{\frac{1}{2}} \left(\sum_{\kappa_2 \in \mathcal{K}_{k_1-k_4}} \|\tilde{P}_{\kappa_2}\psi_2\|^2_{L_t^3 L_x^6} \right)^{\frac{1}{2}} \\
& \quad \cdot \left(\sum_{\kappa_3 \in \mathcal{K}_{k_4}} \|\tilde{P}_{\kappa_3}\psi_3\|^2_{L_t^3 L_x^6} \right)^{\frac{1}{2}} \left(\sum_{\kappa_4 \in \mathcal{K}_{k_4}} \|\tilde{P}_{\kappa_4}\psi_4\|^2_{L_t^\infty L_x^2} \right)^{\frac{1}{2}} \\
& \lesssim 2^{\frac{2(k_1+k_2+k_3)}{3}-k_1} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\
& \lesssim 2^{\frac{2}{3}(k_4-k_1)} J
\end{aligned}
$$

which suffices.

Case 4: there are exactly two $i \in \{1,2,3\}$ such that $k_4 \leq k_i + 10$.

Case 4 a) Assume that $k_3 \geq k_4 - 10$. Since the argument is symmetric in k_1 and k_2, it is enough to consider the scenario $k_1 < k_4 - 10 \leq k_2$. Note that $|k_2 - k_3| \leq 12$.

To streamline the argument we ignore for a moment that in the case $|k_3 - k_4| \leq 9$ the proof below does not cover the estimate for I_{par}. We will explain at the end how to estimate this term.

We claim that either the angle of interactions in $\langle \psi_3, \beta\psi_4 \rangle$ is $\lesssim 2^{\frac{k_1-k_4}{16}}$ or at least one factor $\psi_j, j = 1, \dots, 4$ has modulation $\gtrsim 2^{\frac{k_1+7k_4}{8}}$. To see this, suppose that the claim is false. Then, the modulation of $\langle \psi_1, \beta\psi_2 \rangle$ is $\lesssim 2^{\frac{k_1+7k_4}{8}}$ while it follows from part i) of Lemma 6.5 that the modulation of $\langle \psi_3, \beta\psi_4 \rangle$ is $\gg 2^{\frac{k_1+7k_4}{8}}$. This is not possible, hence the claim is true. Note that in using Lemma 6.5 we are assuming that $k_3, k_4 \geq 100$. If this is not the case, that is $k_3 = 99$, then $k_1, k_2, k_3, k_4 \leq 200$ and this is covered under Case 1.

In the first subcase, where the angle of interaction in $\langle \psi_3, \beta\psi_4 \rangle$ is smaller than $2^{\frac{k_1-k_4}{16}}$, we use (5.1) and (5.2) to estimate

$$
I \lesssim 2^{\frac{k_1-k_4}{16}} 2^{\frac{k_1}{2}} 2^{\frac{k_3}{2}} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \lesssim 2^{\frac{k_1-k_4}{16}} 2^{\frac{k_4-k_2}{2}} J
$$

which is acceptable.

We now consider the second subcase, in which the modulation of the factor ψ_j is $\gtrsim 2^{\frac{k_1+7k_4}{8}} \gtrsim 2^{\frac{k_1+k_4}{2}}$ for some $j \in \{1,2,3,4\}$.

$j=1$: Since ψ_1 has modulation $\gtrsim 2^{\frac{k_1+k_4}{2}}$, we use the Sobolev embedding for ψ_1 to obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^2_t L^\infty_x} \|\psi_2\|_{L^\infty_t L^2_x} 2^{\frac{k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{k_1} \|\psi_1\|_{L^2} \|\psi_2\|_{S_{k_2}} 2^{\frac{k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{k_1-k_4}{4}} 2^{\frac{k_4-k_2}{2}} J. \end{aligned}$$

$j=2$: Since ψ_2 has modulation $\gtrsim 2^{\frac{k_1+k_4}{2}}$, Sobolev embedding for ψ_1 and (5.1) yields

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^\infty} \|\psi_2\|_{L^2} 2^{\frac{k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{k_1} \|\psi_1\|_{L^\infty_t L^2_x} 2^{-\frac{k_1+k_4}{4}} \|\psi_2\|_{S_{k_2}} 2^{\frac{k_3}{2}} \|\psi_3\|_{L^4} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{k_1-k_4}{4}} 2^{\frac{k_4-k_2}{2}} J. \end{aligned}$$

$j=3$: We use (5.6) and estimate as follows

$$\begin{aligned} I &\lesssim \|\langle \psi_1, \beta\psi_2\rangle\|_{L_t^{\frac{p}{p-1}} L^2_x} \|\psi_3\|_{L^p_t L^2_x} \|\psi_4\|_{L^\infty} \\ &\lesssim 2^{\frac{k_1}{p}} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{(1-\frac{1}{p})k_3} 2^{-\frac{k_1+7k_4}{8}} \|\psi_3\|_{S_{k_3}} 2^{k_4} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{(\frac{1}{p}-\frac{5}{8})(k_1-k_3)} 2^{\frac{5k_4-4k_2-k_3}{8}} J. \end{aligned}$$

which is acceptable provided we choose a $\frac{4}{3} < p < \frac{8}{5}$.

$j=4$: We (5.6) and estimate as follows:

$$\begin{aligned} I &\lesssim \|\langle \psi_1, \beta\psi_2\rangle\|_{L^r_t L^2_x} \|\psi_3\|_{L_t^{\frac{2r}{r-2}} L^\infty_x} \|\psi_4\|_{L^2} \\ &\lesssim 2^{(1-\frac{1}{r})k_1} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{(\frac{1}{2}+\frac{1}{r})k_3} \|\psi_3\|_{S_{k_3}} 2^{-\frac{k_1+7k_4}{16}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{(\frac{7}{16}-\frac{1}{r})(k_1-k_3)} 2^{\frac{k_4-k_3}{16}} J \end{aligned}$$

and this is acceptable provided we pick $4 > r > \frac{16}{7}$.

The argument is complete, except that we owe an estimate for I_{par} in the case $|k_3-k_4| \le 9$. Note that, in this case we also have $k_2 \le k_4+15$. By recombining ψ_1 with ψ_4, ψ_2 with ψ_3 (at the cost of having no null structure) and using (5.3), we estimate

$$I_{par} \lesssim 2^{-k_4} 2^{k_1} \|\psi_{k_1}\|_{S_{k_1}} \|\psi_{k_4}\|_{S^w_{k_4}} 2^{k_2} \|\psi_{k_2}\|_{S_{k_2}} \|\psi_{k_3}\|_{S_{k_3}} \lesssim 2^{\frac{k_1-k_4}{2}} J.$$

Case 4 b) Assume now that $k_3 \le k_4 - 10$, hence $k_1, k_2 \ge k_4 - 10$ and $|k_1 - k_2| \le 12$. Here we claim that either the angle of interactions

in $\langle \psi_1, \beta\psi_2\rangle$ is $\lesssim 2^{\frac{k_3-k_4}{16}} 2^{k_4-k_2}$ or at least one factor $\psi_j, j = 1,\ldots,4$ has modulation $\gtrsim 2^{\frac{k_3+7k_4}{8}}$. Indeed, if the claim is false, it follows from Lemma 6.5, part i), that the modulation of $\langle \psi_1, \beta\psi_2\rangle$ is $\gg 2^{\frac{k_3+7k_4}{8}}$ while the modulation of $\langle \psi_3, \beta\psi_4\rangle$ is $\ll 2^{\frac{k_3+7k_4}{8}}$. This is not possible, hence the claim is true. Note that in using Lemma 6.5 we are assuming that $k_1, k_2 \geq 100$. If this is not the case, that is either $k_1 = 99$ or $k_2 = 99$, then $k_1, k_2, k_3, k_4 \leq 200$ the argument is provided in Case 1.

In the first subcase the angle of interaction in $\langle \psi_1, \beta\psi_2\rangle$ is smaller than $2^{\frac{k_3-k_4}{16}} 2^{k_4-k_2}$. Then, we use (5.2) to estimate the contribution of $\langle \psi_1, \beta\psi_2\rangle$ and (5.1) to estimate the contribution of $\langle \psi_3, \beta\psi_4\rangle$. This gives $I \lesssim 2^{\frac{k_3-k_4}{16}} 2^{k_4-k_2} J$ which is acceptable.

In the second subcase, where at least one modulation is high, one proceeds in a similar manner to Case 2a) above. We indicate the starting point in each case and leave the details to the reader.

$j = 1$: We proceed as in the case $j = 4$, Case 2a):

$$I \lesssim \|\psi_1\|_{L^2} \|\psi_2\|_{L_t^{\frac{2p}{p-2}} L_x^\infty} \|\langle \psi_3, \beta\psi_4\rangle\|_{L_t^p L_x^2}.$$

$j = 2$: Identical to the case $j = 1$.

$j = 3$: We proceed as in the case $j = 1$, Case 2a):

$$I \lesssim 2^{\frac{k_1}{2}} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} \|\psi_3\|_{L_t^2 L_x^\infty} \|\psi_4\|_{L_t^\infty L_x^2}.$$

$j = 4$: We proceed as in the case $j = 2$, Case 2b):

$$I \lesssim 2^{\frac{k_1}{2}} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} \|\psi_3\|_{L^\infty} \|\psi_4\|_{L^2}.$$

<u>Case 5: $|k_2 - k_4| \leq 2$ and $k_1, k_3 \leq k_4 - 10$.</u> Without restricting the generality of the argument, we may assume that $k_1 \leq k_3$.

We claim that either the angle of interaction in $\langle \psi_3, \beta\psi_4\rangle$ is $\lesssim 2^{\frac{k_1-k_3}{16}}$ or one factor $\psi_j, j = 1,\ldots,4$ has modulation $\gtrsim 2^{\frac{k_1+7k_3}{8}}$. Indeed, if all modulations of the functions involved are $\ll 2^{\frac{k_1+7k_3}{8}}$, then $\langle \psi_1, \beta\psi_2\rangle$ is localized at modulation $\lesssim 2^{\frac{k_1+7k_3}{8}}$. This forces $\langle \psi_3, \beta\psi_4\rangle$ to be localized at modulation $\lesssim 2^{\frac{k_1+7k_3}{8}}$, hence the angle of interaction is $\lesssim 2^{\frac{k_1-k_3}{16}}$ by Lemma 6.5, part i). Note that in using Lemma 6.5 we are assuming that $k_3, k_4 \geq 100$. If this is not the case, that is $k_3 = 99$, then $k_1 = 99$ and the estimate $I \lesssim J$ suffices.

In the first subcase, when the angle of interaction in $\langle \psi_3, \beta\psi_4\rangle$ is $\lesssim 2^{\frac{k_1-k_3}{16}}$, we use (5.2) to obtain

$$I \lesssim 2^{\frac{k_1-k_3}{16}} 2^{\frac{k_1}{2}} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{\frac{k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \lesssim 2^{\frac{k_1-k_3}{16}} J.$$

Next, we consider the second subcase when the factor ψ_j has modulation $\gtrsim 2^{\frac{k_1+7k_3}{8}} \gtrsim 2^{\frac{k_1+3k_3}{4}}$ for some $j \in \{1,2,3,4\}$:

$j=1$: The modulation of ψ_1 is $\gtrsim 2^{\frac{k_1+3k_3}{4}}$, so we use Sobolev embedding for ψ_1 and (5.1) for $\langle \psi_3, \beta\psi_4\rangle$ to obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^2_t L^\infty_x} \|\psi_2\|_{L^\infty_t L^2_x} 2^{\frac{k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{k_1} \|\psi_1\|_{L^2} \|\psi_2\|_{S_{k_2}} 2^{\frac{k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{k_1} 2^{-\frac{k_1+3k_3}{8}} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{\frac{k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{k_1-k_3}{8}} J. \end{aligned}$$

$j=2$: Here, the modulation of ψ_2 is $\gtrsim 2^{\frac{k_1+3k_3}{4}}$ and we proceed as above to obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^\infty} \|\psi_2\|_{L^2} 2^{\frac{k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{k_1} \|\psi_1\|_{L^\infty_t L^2_x} 2^{-\frac{k_1+3k_3}{8}} \|\psi_2\|_{S_{k_2}} 2^{\frac{k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{k_1-k_3}{8}} J. \end{aligned}$$

$j=3$: The modulation of ψ_3 is $\gtrsim 2^{\frac{k_1+7k_3}{8}}$, we use the Sobolev embedding for ψ_3 to obtain

$$\begin{aligned} I &\lesssim \|\langle \psi_1, \beta\psi_2\rangle\|_{L_t^{\frac{p}{p-1}} L^2_x} \|\psi_3\|_{L^p_t L^\infty_x} \|\psi_4\|_{L^\infty_t L^2_x} \\ &\lesssim \|\langle \psi_1, \beta\psi_2\rangle\|_{L_t^{\frac{p}{p-1}} L^2_x} 2^{k_3} \|\psi_3\|_{L^p_t L^2_x} \|\psi_4\|_{L^\infty_t L^2_x} \\ &\lesssim 2^{\frac{k_1}{p}} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{(2-\frac{1}{p})k_3} 2^{-\frac{k_1+7k_3}{8}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{(\frac{1}{p}-\frac{5}{8})(k_1-k_3)} J. \end{aligned}$$

which is acceptable provided we choose a $\frac{4}{3} < p < \frac{8}{5}$.

$j=4$: Since the modulation of ψ_4 is $\gtrsim 2^{\frac{k_1+7k_3}{8}}$, we estimate as follows

$$\begin{aligned} I &\lesssim \|\langle \psi_1, \beta\psi_2\rangle\|_{L^p_t L^2_x} \|\psi_3\|_{L_t^{\frac{2p}{p-2}} L^\infty_x} \|\psi_4\|_{L^2} \\ &\lesssim 2^{(1-\frac{1}{p})k_1} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{(\frac{1}{2}+\frac{1}{p})k_3} \|\psi_3\|_{S_{k_3}} 2^{-\frac{k_1+7k_3}{16}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{(\frac{7}{16}-\frac{1}{p})(k_1-k_3)} J \end{aligned}$$

and this is acceptable provided we pick $4 > p > \frac{16}{7}$.

<u>Case 6: $|k_1-k_4| \le 2$ and $k_2, k_3 \le k_4 - 10$.</u> By switching the roles of ψ_1 and ψ_2, this case is entirely similar to Case 5.

Case 7: $|k_3 - k_4| \leq 2$ and $k_1, k_2 \leq k_4 - 10$. Without loss of generality we assume $k_1 \leq k_2$. Since $|k_3 - k_4| \leq 2$ there will be a problem with estimating I_{par}. We estimate this term the same way we did in Case 3 (see $k_1, k_2 \leq k_4 + 15$ part there) to obtain: $I_{par} \lesssim 2^{\frac{k_1+k_2-2k_4}{6}} J$ and this is fine. As a consequence, in the rest of the argument we can tacitly ignore that the estimates we provide do not work for the I_{par} part of I.

The key observation is that either the angle of interaction between ψ_3 and ψ_4 is $\lesssim 2^{\frac{k_1-k_2}{16}} 2^{k_2-k_3}$ or at least one factor has modulation $\gtrsim 2^{\frac{k_1+7k_2}{8}}$. Indeed, if all modulations are $\ll 2^{\frac{k_1+7k_2}{8}}$, then the modulation of $\langle \psi_1, \beta\psi_2 \rangle$ is $\lesssim 2^{\frac{k_1+7k_2}{8}}$ and part i) of Lemma 6.5 implies the claim. Note that in using Lemma 6.5 we are assuming that $k_3, k_4 \geq 100$. If this is not the case, that is $k_3 = 99$, then $k_1, k_2, k_3, k_4 \leq 200$ and the argument is provided in Case 1.

We consider the first subcase, when the angle of interaction between ψ_3 and ψ_4 is $\lesssim 2^{\frac{k_1-k_2}{16}} 2^{k_2-k_3}$. Using (5.1) and (5.2) we estimate

$$I \lesssim 2^{\frac{k_1}{2}} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{\frac{k_3}{2}} 2^{\frac{k_1-k_2}{32}} 2^{\frac{k_2-k_3}{2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \lesssim 2^{\frac{k_1-k_2}{32}} J.$$

In the second subcase, ψ_j has modulation $\gtrsim 2^{\frac{k_1+7k_2}{8}}$ for some $j \in \{1, 2, 3, 4\}$.

$j = 1$: The modulation of ψ_1 is $\gtrsim 2^{\frac{k_1+7k_2}{8}}$. Using (5.10) with $p = 2$ for ψ_2, ψ_3, ψ_4, and the Sobolev embedding for ψ_1 we estimate

$$\begin{aligned}
I &\lesssim \|\psi_1\|_{L^2_t L^\infty_x} 2^{\frac{k_2}{2}} \|\psi_2\|_{S_{k_2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\
&\lesssim 2^{k_1} \|\psi_1\|_{L^2} 2^{\frac{k_2}{2}} \|\psi_2\|_{S_{k_2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\
&\lesssim 2^{k_1} 2^{-\frac{k_1+7k_2}{16}} \|\psi_1\|_{S_{k_1}} 2^{\frac{k_2}{2}} \|\psi_2\|_{S_{k_2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\
&\lesssim 2^{\frac{7}{16}(k_1-k_2)} J.
\end{aligned}$$

$j = 2$: Using (5.10) for ψ_1, ψ_3, ψ_4 and the Sobolev embedding for ψ_2 we proceed as follows:

$$\begin{aligned}
I &\lesssim 2^{(1-\frac{1}{q})k_1} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{L_t^{\frac{q}{q-1}} L^\infty_x} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\
&\lesssim 2^{(1-\frac{1}{q})k_1} \|\psi_1\|_{S_{k_1}} 2^{k_2} \|\psi_2\|_{L_t^{\frac{q}{q-1}} L^2_x} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\
&\lesssim 2^{(1-\frac{1}{q})k_1} \|\psi_1\|_{S_{k_1}} 2^{k_2} 2^{\frac{k_2}{q}} 2^{-\frac{k_1+7k_2}{8}} \|\psi_2\|_{S_{k_2}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\
&\lesssim 2^{(\frac{3}{8}-\frac{1}{q})(k_1-k_2)} J.
\end{aligned}$$

which is acceptable as long as $p = \frac{q}{q-1} \in (\frac{4}{3}, \frac{8}{5})$ and $\frac{1}{q} < \frac{3}{8}$, which is both satisfied as long as $\frac{8}{3} < q < 4$.

$j = 3$ and $j = 4$: Here we assume that ψ_3 and ψ_4 have modulation $\gtrsim 2^{\frac{k_1+7k_2}{8}}$. In this case we estimate

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^\infty}\|\psi_2\|_{L^\infty}\|\psi_3\|_{L^2}\|\psi_4\|_{L^2} \\ &\lesssim 2^{k_1+k_2}\|\psi_1\|_{S_{k_1}}\|\psi_2\|_{S_{k_2}} 2^{-\frac{k_1+7k_2}{8}}\|\psi_3\|_{S_{k_3}}\|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{3}{8}(k_1-k_2)} J. \end{aligned}$$

$j = 3$ (only): The modulation of ψ_3 is $\gtrsim 2^{\frac{k_1+7k_2}{8}}$ and all the other terms have modulation $\ll 2^{\frac{k_1+7k_2}{8}}$. In this case we note that the angle of interaction between ψ_2 and ψ_4 is $\gtrsim 2^{\frac{k_1-k_2}{16}}$ or else their interaction has modulation $\ll 2^{\frac{k_1+7k_2}{8}}$ and this cannot be changed by ψ_1 to match the modulation of ψ_3. Thus combine ψ_2 and ψ_4, use (5.3) to obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^\infty} 2^{\frac{k_2}{2}} 2^{-\frac{k_1-k_2}{32}}\|\psi_2\|_{S_{k_2}}\|\psi_4\|_{S_{k_4}}\|\psi_3\|_{L^2} \\ &\lesssim 2^{k_1}\|\psi_1\|_{S_{k_1}} 2^{\frac{k_2}{2}} 2^{-\frac{k_1-k_2}{32}}\|\psi_2\|_{S_{k_2}} 2^{-\frac{k_1+7k_2}{16}}\|\psi_3\|_{S_{k_3}}\|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{(\frac{7}{16}-\frac{1}{32})(k_1-k_2)} J. \end{aligned}$$

$j = 4$ (only): We change the role of ψ_3 and ψ_4 in the above argument.

We are now done with the analysis of (6.5) in the case $s_1 = s_2 = s_3 = s_4 = +$. It is obvious that the same argument works for $s_1 = s_2 = s_3 = s_4 = -$. Next we indicate how the other sign choices can be dealt with, by highlighting the similarities and differences from the choice $s_1 = s_2 = s_3 = s_4 = +$. We do this by going over each case.

No changes are needed in the easy cases: Case 1 and Case 2.

<u>Case 3: $k_4 \le \min(k_1, k_2, k_3) + 10$.</u> Here the only part that needs to be adjusted is the last scenario when $k_4 - 10 \le k_3 \le k_4 + 9, k_1 > k_4 + 15, k_2 > k_4 + 10$ and $s_1 = -s_2$. As already argued there, only the interactions between ψ_1 and ψ_2 making an angle $\lesssim 2^{k_4-k_1}$ have nontrivial contribution to I, that is only pairs $\langle \tilde{P}_{\kappa_1}\psi_1, \beta \tilde{P}_{\kappa_2}\psi_2\rangle$ with $d(\kappa_1, \kappa_2) \lesssim 2^{k_4-k_1}$. But this implies $d(\kappa_1, -\kappa_2) \approx 1$, and we claim that at least one factor has modulation $\gtrsim 2^{k_1}$. Indeed, otherwise all factors have modulations $\ll 2^{k_1}$ from which we obtain two contradictory results: $\langle \psi_1, \beta\psi_2\rangle$ has modulation $\approx 2^{k_1}$ (on behalf of part ii) of Lemma 6.5) while $\langle \psi_3, \beta\psi_4\rangle$ has modulation $\ll 2^{k_1}$.

Now it is an easy exercise to establish the desired estimate, given that at least one factor has modulation $\gtrsim 2^{k_1}$.

<u>Case 4: there are exactly two $i \in \{1, 2, 3\}$ such that $k_4 \le k_i + 10$.</u>

Case 4 a) Assume that $k_3 \ge k_4 - 10$. The argument is the same

if $s_3 = s_4$. If $s_3 = -s_4$ then the new claim is: either the angle of interactions in $\langle \psi_3, \beta\psi_4\rangle$ is $\pi + \alpha$ with $|\alpha| \lesssim 2^{\frac{k_1-k_4}{16}}$ or at least one factor $\psi_j, j = 1, \dots, 4$ has modulation $\gtrsim 2^{\frac{k_1+7k_4}{8}}$. This claim is proved in a similar manner, just that now we invoke part ii) of Lemma 6.5. Then the rest of the argument is carried in a similar manner.

Case 4 b) Assume that $k_3 \leq k_4 - 10$, hence $k_1, k_2 \geq k_4 - 10$ and $|k_1 - k_2| \leq 12$. If $s_1 = s_2$ the proof is the same.

If $s_1 = -s_2$ and $k_1, k_2 \leq k_4 + 10$, then the claim there is modified as follows: either the angle of interactions in $\langle \psi_1, \beta\psi_2\rangle$ is $\pi + \alpha$ with $|\alpha| \lesssim 2^{\frac{k_3-k_4}{16}}$ or at least one factor $\psi_j, j = 1, \dots, 4$ has modulation $\gtrsim 2^{\frac{k_3+7k_4}{8}}$. This is proved using part ii) of Lemma 6.5. Then the rest of the argument follows in a similar manner.

If $s_1 = -s_2$ and $\max(k_1, k_2) \geq k_4 + 11$, in which case $k_1, k_4 \geq k_4 + 6$, then only interactions at angle $\lesssim 1$ in $\langle \psi_1, \beta\psi_2\rangle$ contribute to I given that the output $\langle \psi_1, \beta\psi_2\rangle$ is localized at much lower frequency. Using part ii) of Lemma 6.5 we conclude that at least one factor ψ_j has modulation $\gtrsim 2^{k_1}$ and then the argument becomes easier.

Case 5: $|k_2 - k_4| \leq 2$ and $k_1, k_3 \leq k_4 - 10$. Without restricting the generality of the argument, we may assume that $k_1 \leq k_3$.

No modification is needed if $s_3 = s_4$. If $s_3 = -s_4$ then the claim is modified to: either the angle of interaction in $\langle \psi_3, \beta\psi_4\rangle$ is $\pi + \alpha$ with $|\alpha| \lesssim 2^{\frac{k_1-k_3}{16}}$ or one factor $\psi_j, j = 1, \dots, 4$ has modulation $\gtrsim 2^{\frac{k_1+7k_3}{8}}$. This is done using part ii) of Lemma 6.5. The rest of the argument is similar.

Case 6: $|k_1 - k_4| \leq 2$ and $k_2, k_3 \leq k_4 - 10$. By switching the roles of ψ_1 and ψ_2, this case is entirely similar to Case 5.

Case 7: $|k_3 - k_4| \leq 2$ and $k_1, k_2 \leq k_4 - 10$. Without loss of generality we assume $k_1 \leq k_2$. No modification is needed if $s_3 = s_4$. If $s_3 = -s_4$ then only interactions at angle $\lesssim 1$ in $\langle \psi_3, \beta\psi_4\rangle$ contribute to I given that the output $\langle \psi_3, \beta\psi_4\rangle$ is localized at much lower frequency. Using part ii) of Lemma 6.5 we conclude that at least one factor ψ_j has modulation $\gtrsim 2^{k_4}$ and then the argument becomes easier. □

Based on Theorem 6.1 we can now prove Theorem 1.1 concerning the global well-posedness and scattering of the cubic Dirac equation for small data.

Proof of Theorem 1.1. In Section 3 we reduced the study of the cubic Dirac equation to the study of the system (3.3). In the nonlinearity of (3.3) we split the functions into $\psi = \psi_+ + \psi_-$ where $\psi_\pm = \Pi_\pm\psi$ and note that $\psi_\pm = \Pi_\pm\psi_\pm$. Using the nonlinear estimate in Theorem 6.1 and the

linear estimates in Corollary 4.5, a standard fixed point argument in a small ball in the space $S_C^{+,\frac{1}{2}}(I) \times S_C^{-,\frac{1}{2}}(I)$ gives local existence on every time interval I containing 0, uniqueness and Lipschitz continuity of the flow map for small initial data $(\psi_+(0), \psi_-(0)) \in H^{\frac{1}{2}}(\mathbb{R}^2) \times H^{\frac{1}{2}}(\mathbb{R}^2)$. Since all the bounds are independent on the size of I, this implies global existence, uniqueness and Lipschitz continuity of the flow map for small initial data $(\psi_+(0), \psi_-(0)) \in H^{\frac{1}{2}}(\mathbb{R}^2) \times H^{\frac{1}{2}}(\mathbb{R}^2)$.

Concerning scattering, we simply use the fact that $\psi_\pm \in V_\pm^2 H^{\frac{1}{2}}$: this is obtained first on every time interval I with bounds independent of the size of I which then implies the global bound on $\mathbb{R}$. □

Acknowledgement. The first author was supported in part by NSF grant DMS-1001676. The second author acknowledges support from the German Research Foundation, Collaborative Research Center 701.

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